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Random response of hysteretic systems with viscous damping to white excitations

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ABSTRACT: The nonlinear random response of the single-degree-of-freedom system having the elasto-plastic hysteretic restoring force together with the viscous damping is presented. The Gaussian white noise is assumed as the ground acceleration. Two types of damping where the coefficient of viscousity is taken proportional to the elastic stiffness and to the instantaneous stiffness are dealt with. The analysis aims at finding the approximate solutions for the expectation and the variance of displacement, velocity, ductility factor, cumulative ductility factor and damping energy. The solutions are compared and well agree with the digital estimates in most cases.

1. INTROCUTION

The nonlinear random response of the undamped single-degree-of-freedom system with bilinear hysteretic characteristic was analysed by Matsushima(1990). The system dealt with herein is identical with that in his analysis except that it has the dashpot where the damping force is in proportion to the velocity. Two types of damping where the coefficient of viscousity is taken proportional to the elastic stiffness and to the instantaneous stiffness are assumed. The fundamental idea of the analysis is same as in the undamped case. The displacement is decomposed into two components ---the shift of the center of oscillation and the deformation about its displaced center. The former is caused by the yield excursion which has the random positive or negative magnitude and regarded as the diffusion process. The theory of the one-dimensional random walk is therefore available for its evaluation. The latter is estimated by taking account of the equivalent natural frequency of the oscillating portion together with the concept of energy balance. The analysis aims at finding the approximate solutions for the important response quantities which are needed for the seismic design of structures. The digital simulation is performed in order to verify the accuracy of approximate solutions. The effects of magnitude and type of damping on responses are examined.

2. INPUT-OUTPUT SYSTEM

The structure is idealized by the mass-dashpot-spring system having single degree of freedom. Two types of damping where the viscousity is taken proportional to the elastic stiffness and to the instantaneous stiffness are dealt with. The viscous damping coefficient c is represented by $(2h/\omega_0)k_0 = 2h\omega_0 m$ in the former case, where h is the damping ratio, $k_0(=\omega_0^2 m)$ is the elastic stiffness, ω_0 is the natural angular frequency in the elastic region and m is the mass. This case is simply referred to as the linear damping hereafter. In the latter case c is represented by $(2h/\omega_0)k_t$, where k_t is the instantaneous stiffness. This case is referred to as the nonlinear damping.

The system rested on the ground is suddenly subjected to the Gaussian white noise which is taken as the ground acceleration. The equation of motion is given by

$$\ddot{x}+2h\omega_t\dot{x}+f(x,\dot{x})=-N(t), \qquad (1)$$

where x is the displacement of the system and · means the derivative with respect to time t. N(t) denotes the stationary white noise having zero mean and the constant spectral density S_0. $f(x,\dot{x})$ represents a restoring force function which has the bilinear hysteretic characteristic as illustrated in Fig.1. The perfect plastic flow takes place at the yield acceleration $\alpha(=\omega_0^2\Delta)$, where Δ is the yield displacement. ω_t is always ω_0 in the case of linear damping. ω_t is either ω_0 or 0 in the case of nonlinear damping, depending on two slope angles in the hysteresis.

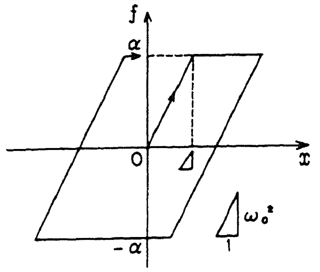

Fig.1 Bilinear hysteresis with zero plastic stiffness

3. TIME FOR RESPONSE TO ARRIVE AT ELASTIC LIMIT

The initial conditions of Eq.(1) are $x=\dot{x}=0$ when $t=0$. In the nonstationary response, the system behaves elastically in the early stage. The expected time for the response to arrive at the elastic limit, which is denoted by t_0, is approximately estimated in the following:

The expectation of the maximum absolute elastic displacement is given by

$$|\bar{\eta}|_{max}=a\sigma_{\eta}, \tag{2}$$

where η designates x/Δ. The symbol bar and σ mean the expectation and the standard deviation. The value of a becomes in an approximate sense

$$a=\sqrt{2}\,f(h\tau), \tag{3}$$

where

$$f(h\tau)=\sqrt{0.424+\ln(4\pi h\tau+1.78)}. \tag{4}$$

τ denotes the time normalized by the natural period of the system T_0. The function given by Eq.(4) was proposed by Rosenblueth et al. (1962) as the ratio of the peak factor of damped response to that of undamped one.

σ_{η} is approximately expressed by

$$\sigma_{\eta}=\sqrt{\frac{\pi\xi}{2h}\left(1-e^{-4\pi h\tau}\right)}, \tag{5}$$

where $\xi \equiv \omega_0 S_0/\alpha^2$. $\tau_0(\equiv t_0/T_0)$ is obtained as the time which satisfies $a\sigma_{\eta}=1$. The repetition of calculation is needed to get the solution.

4. RESPONSE IN ELASTO-PLASTIC REGION

The system goes into the plastic region when t is greater than t_0. The energy supplied to the system soon balances the energy dissipated by the viscous damping and the hysteresis. The displacement, however, never arrives at the stationary stage. It is approximately decomposed into two components as

$$x=x_c+x_0, \tag{6}$$

where x_c and x_0 represent the shift of the center of oscillation and the deformation about its displaced center. x_c has the very low frequency, whereas x_0 has the equivalent angular frequency ω_e which is independent of time. The energy driven to the system is consumed by the viscous damping and by the hysteresis loop which is related to x_0.

The oscillating component about the center of hysteresis loop, x_0, is approximately sinusoidal with the expected equivalent angular frequency ω_e. The energy balance during the equivalent period $T_e(=2\pi/\omega_e)$ gives

$$\bar{x}_{p1}\alpha+2h\omega_0 V_{\dot{x}}\frac{2\pi}{\omega_e}=\pi S_0\frac{2\pi}{\omega_e}, \tag{7}$$

in the case of linear damping. $\bar{x}_{p1}$ denotes the expectation of cumulative plastic deformation per T_e. The plastic deformation means herein the deformation which excurses on the lines $f=\pm\alpha$. The cumulative plastic deformation is defined by the sum of the absolute values of plastic deformation.

$\dot{x}$ is approximately equal to $\dot{x}_0$ as found from Eq.(6), since $\dot{x}_c$ is much less than $\dot{x}_0$. $\dot{x}_0$ is assumed to vibrate sinusoidally with the amplitude $\omega_e(\bar{x}_{p1}/4+\Delta)$. Supposed that the hysteresis loop is symmetric about its center and that the slope of its diagonal corresponds to ω_e^2, then

$$\omega_e^2=\frac{4\alpha}{\bar{x}_{p1}+4\Delta}. \tag{8}$$

Equation (8) is rewritten in the nondimenional form by introducing $\beta(\equiv\omega_e/\omega_0)$ and $\bar{\lambda}_1(\equiv\bar{x}_{p1}/\Delta)$ as

$$\beta^2=\frac{4}{\bar{\lambda}_1+4}. \tag{9}$$

It is possible to write the variance of $\dot{x}$ as

$$V_{\dot{x}}=\frac{\omega_e^2}{a_0^2}\left(\frac{\bar{x}_{p1}}{4}+\Delta\right)^2, \tag{10}$$

where a_0 is estimated from

$$a_0=\sqrt{2}\,f(h\tau_0), \tag{11}$$

considering the continuety of equation at the time τ_0.

Substituting $V_{\dot{x}}$ given by Eq.(10) into Eq.(7)

and expressing all quantities in the nondimensional forms, one gets

$$\bar{\lambda}_1 = \frac{2\pi}{\beta^3}\left(\pi\beta^2\xi - \frac{2h}{a_0^2}\right). \quad (12)$$

$\bar{\lambda}_1$ and β are obtained by solving Eqs.(9) and (12) simultaneously. The repetition of calculation is needed to get the solutions.

The ratio of the coefficient of linear damping to that of nonlinear one is $\omega_e^2/\omega_0^2(=\beta^2)$ on the average. $h\beta^2$ is used, therefore, instead of h appearing in Eq.(12) in the nonlinear damping case in the approximate sense. It is found from the numerical calculation that β does not depend so much on h and on the type of damping.

Equation (10) is rewritten by the aid of Eq.(9) together with the introduction of $\dot{\eta}\,(\equiv \dot{x}/(\omega_0\Delta))$ in the following nondimensional form:

$$V_{\dot{\eta}} = \frac{1}{a_0^2\beta^2} \quad (13)$$

Incidentally Eq.(12) can be written as

$$\bar{\lambda}_1 = \frac{4\pi h}{\beta}\left(\frac{\pi\xi}{2h} - V_{\dot{\eta}}\right), \quad (14)$$

by using Eq.(13). This expression shows that the system is always in the elasto-plastic region when $\tau'(\equiv \tau-\tau_0)>0$, since Eq.(5) indicates that $V_{\dot{\eta}} < \pi\xi/(2h)$ when the system is on the elastic limit where $\tau'=0$ and $\beta=1$.

The hysteretic damping can be replaced by the equivalent viscous damping as

$$\bar{x}_{p1}\alpha = 2h_e\omega_e V_{\dot{x}}\frac{2\pi}{\omega_e}, \quad (15)$$

where h_e is the equivalent damping ratio. The reduction of Eq.(15) to the nondimensional form leads to

$$h_e = \frac{a_0^2\beta^2\bar{\lambda}_1}{4\pi}. \quad (16)$$

This expression is used hereafter in the estimation of peak factor p.

5. CUMULATIVE DUCTILITY FACTOR AND DAMPING ENERGY

The expectation of cumulative plastic deformation x_a, when $t' \geqq 0$, becomes

$$\bar{x}_a = \bar{x}_{p1}t'\frac{\omega_e}{2\pi}. \quad (17)$$

The nondimensional form of Eq.(17) is given by

$$\bar{\lambda} = \bar{\lambda}_1\beta\tau', \quad (18)$$

where $\lambda(\equiv x_a/\Delta)$ is referred to as the cumulative ductility factor.

The expectation of nondimensional expression of the viscously dissipated energy D, which is defined by $\delta \equiv D/(\alpha\Delta)$, is equal to the expectation of nondimensional total energy ε, which is the total energy divided by $\alpha\Delta$, subtracted by $\bar{\lambda}$. Since $\bar{\varepsilon}=2\pi^2\xi\tau'$, there is obtained

$$\bar{\delta} = (2\pi^2\xi - \bar{\lambda}_1\beta)\tau'. \quad (19)$$

This is rewritten by the aid of Eqs.(12) and (13) as

$$\bar{\delta} = 4\pi h\tau' V_{\dot{\eta}}. \quad (20)$$

The ratio of hysteretically dissipated energy to the total energy becomes from Eqs.(18) and (20)

$$\frac{\bar{\lambda}}{\bar{\lambda}+\bar{\delta}} = \frac{1}{1+\dfrac{4\pi h V_{\dot{\eta}}}{\bar{\lambda}_1\beta}}, \quad (21)$$

which is independent of τ'. $h\beta^2$ is substituted for h in Eqs.(20) and (21) in the case of nonlinear damping.

Supposed that the velocity response is sinusoidal with the amplitude A and the angular frequency ω_e, the expectation and the mean square value of damping energy per half period D_H are, respectively, $(\pi h\omega_0/\omega_e)\bar{A}^2$ and $(\pi h\omega_0/\omega_e)^2\bar{A}^4$. Therefore the variance of D_H is equal to $(\pi h\omega_0/\omega_e)^2(\bar{A}^4-\bar{A}^{2\,2})$. Under the assumption that the amplitude A is the random variable having the Rayleigh probability density function, the following relationship holds:

$$\bar{A}^4 - \bar{A^2}^2 = 8\sigma_{\dot{x}}^4 - (2\sigma_{\dot{x}}^2)^2 = 4\sigma_{\dot{x}}^2. \quad (22)$$

If it is possible to assume that the damping energies per half period are mutually independent by analogy with the fact that the cumulative plastic deformation with positive velocity is independent of that of negative velocity, the variance of damping energy during the time t' is approximately given by

$$V_D = \frac{4\pi h^2\omega_0^2}{\omega_e}V_{\dot{x}}^2 t'. \quad (23)$$

The nondimensional expression of Eq.(23) is

$$V_\delta = \frac{8\pi^2h^2\tau'}{\beta}V_{\dot{\eta}}^2. \quad (24)$$

$h\beta^2$ is again used instead of h in Eq.(24) in the nonlinear damping case.

It is simply assumed that λ is perfectly correlated to δ, since they seem to be strongly correlated to each other such that both λ and δ increase at the same time as the random variable ε increases. That is

$$\sigma_{\varepsilon} = \sigma_{\lambda} + \sigma_{\delta}. \tag{25}$$

The variance of ε is given by Matsushima (1990) as

$$V_{\varepsilon} = 4\pi^2 \xi \tau' V_{\dot{\eta}}. \tag{26}$$

The application of Eqs.(24) and (26) to Eq.(25) gives

$$\sigma_{\lambda} = 2\pi\sqrt{\tau'}\left(\sqrt{\xi} - h\sigma_{\dot{\eta}}\sqrt{\frac{2}{\beta}}\right)\sigma_{\dot{\eta}} \tag{27}$$

in the linear damping case. Here $h\beta^2$ is substituted for h in the nonlinear damping case.

6. DISPLACEMENT AND DUCTILITY FACTOR

Under the assumption that η_c and η_θ are statistically independent, the variance of η is given by Matsushima (1990) as

$$V_{\eta} = V_{\eta_c} + V_{\eta_\theta} = V_{\lambda} + \frac{V_{\dot{\eta}}}{\beta^2}. \tag{28}$$

In the derivation of Eq.(28), η_c is regarded as the random walk having zero mean and variance V_{λ}. Applying Eq.(27) to Eq.(28), one gets

$$V_{\eta} = \left\{4\pi^2\tau'\left(\sqrt{\xi} - h\sigma_{\dot{\eta}}\sqrt{\frac{2}{\beta}}\right)^2 + \frac{1}{\beta^2}\right\}V_{\dot{\eta}} \tag{29}$$

in the linear damping case. Here $h\beta^2$ is substituted for h in the nonlinear damping case.

In the similar manner, the expectation of ductility factor μ which is defined by $|x|_{max}/\Delta$ approximately becomes

$$\bar{\mu} = \sqrt{\overline{|\eta_c|}^2_{max} + \overline{|\eta_\theta|}^2_{max}} = \sqrt{\frac{\pi}{2}V_{\lambda} + \frac{p^2}{\beta^2}V_{\dot{\eta}}}, \tag{30}$$

where the suffix max means the maximum value. p is the "peak factor", which is estimated from

$$p = \sqrt{2}\, f(h\tau + \beta h_e \tau'), \tag{31}$$

in the linear damping case, considering the

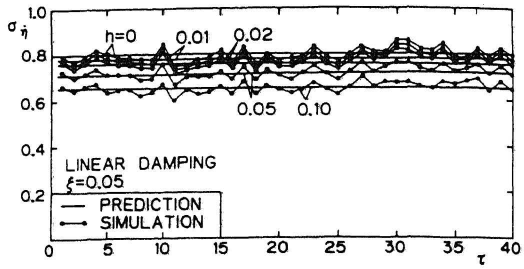

Fig.2 Time change of standard deviation of velocity

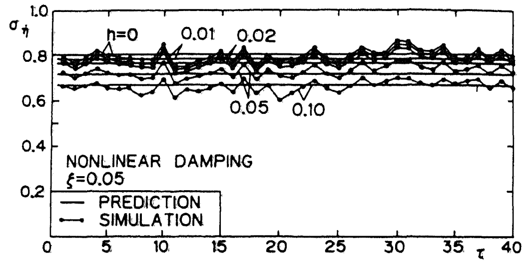

Fig.3 Time change of standard deviation of velocity

continuety of equation at the time τ_θ and the increase of apparent damping after τ_θ. h_e is evaluated from Eq.(16). $h\beta^2$ is again substituted for h in the nonlinear damping case. Application of Eq.(27) to Eq.(30) leads to the following concrete expression for $\bar{\mu}$ in the linear damping case:

$$\bar{\mu} = \sqrt{2\pi^3\tau'\left(\sqrt{\xi} - h\sigma_{\dot{\eta}}\sqrt{\frac{2}{\beta}}\right)^2 + \frac{p^2}{\beta^2}}\;\sigma_{\dot{\eta}} \tag{32}$$

The variance of μ is approximately written by Matsushima (1990) as

$$V_{\mu} = V_{|\eta_c|_{max}} + V_{|\eta_\theta|_{max}} \approx V_{|\eta_c|_{max}}$$
$$= \left(2c - \frac{\pi}{2}\right)V_{\lambda}, \tag{33}$$

where c is Catalan's constant which nearly equals 0.9160.

7. VERIFICATION BY DIGITAL SIMULATION

The digital simulation is performed in order to verify appoximate solutions obtained above. Three hundred white noise samples are

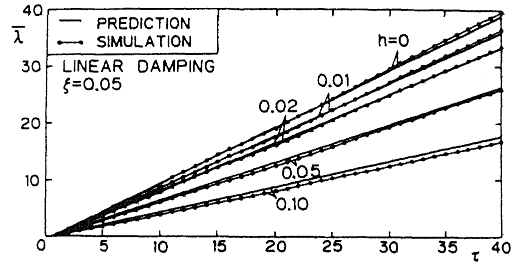

Fig.4 Time change of expectation of cumulative ductility factor

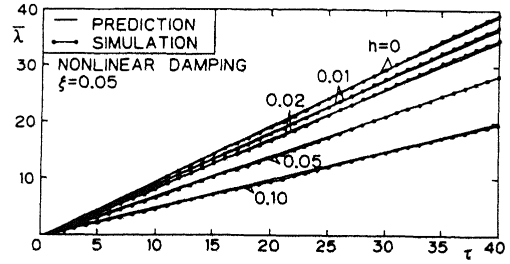

Fig.5 Time change of expectation of cumulative ductility factor

generated. The nonstationary nonlinear responses are numerically computed. The statistical treatment is made on the results. The values of ξ are taken as 0.025, 0.05, 0.075 and 0.1. h is assumed as 0, 0.01, 0.02, 0.05 and 0.10.

$\sigma_{\dot{\eta}} - \tau$ relations with a parameter h are displayed in Figs.2 and 3 for linear and nonlinear damping cases. The solid lines stand for approximate solutions for five different damping ratios. The points connected by fine lines show the simulation estimates. ξ is fixed on 0.05 in these examples. This value corresponds to the ratio of the intensity of quite strong ground motion to the strength of typical low-rise building. For example, let $S_0=200\sim300\ cm^2\cdot s^{-3}$, $T=0.3\sim0.5$ s and $\alpha=300\ cm\cdot s^{-2}$, then ξ turns out to be

$$\xi = \frac{2\pi}{(0.3\sim0.5)} \times \frac{(200\sim300)}{300^2} = 003\sim0.07$$

The value 0.05 is exclusively taken for ξ in the figures shown hereafter as the representative. The solid lines agree well with associated points. The type of damping does not affect $\sigma_{\dot{\eta}}$ so much.

$\bar{\lambda} - \tau$ relations are shown in Figs. 4 and 5 in the same manner as in the previous figures. The degrees of agreement between the both

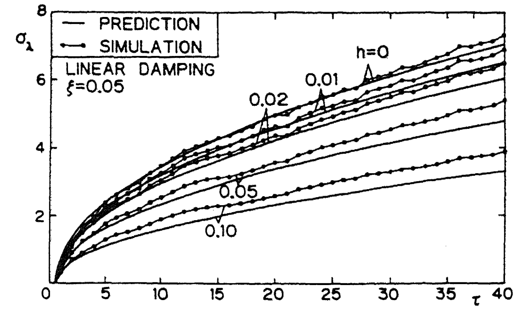

Fig.6 Time change of standard deviation of cumulative ductility factor

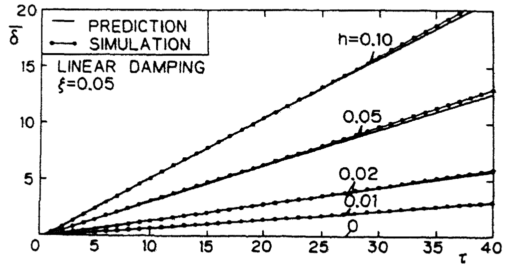

Fig.7 Time change of expectation of damping energy

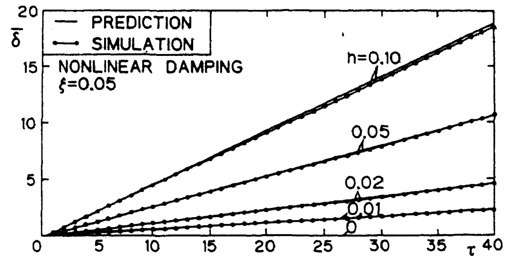

Fig.8 Time change of expectation of damping energy

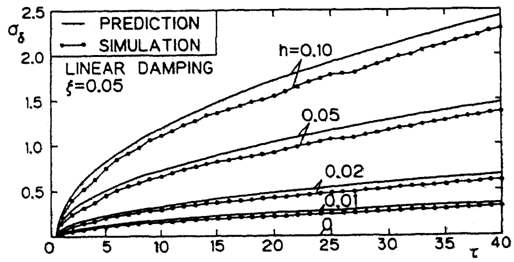

Fig.9 Time change of standard deviation of damping energy

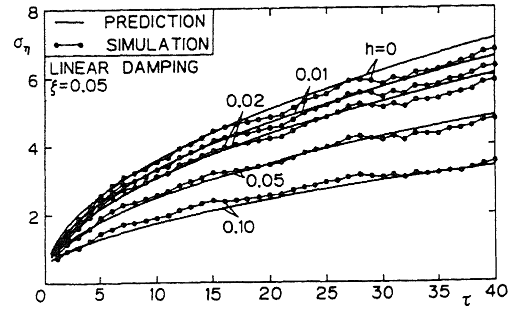

Fig.10 Time change of standard deviation of displacement

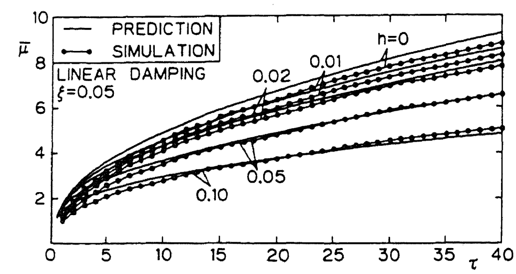

Fig.11 Time change of expectation of ductility factor

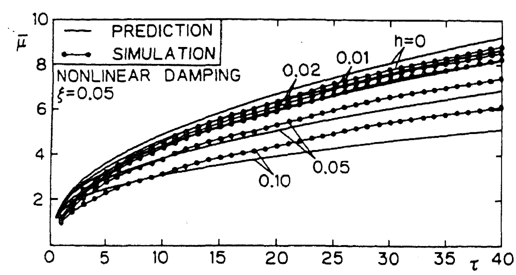

Fig.12 Time change of expectation of ductility factor

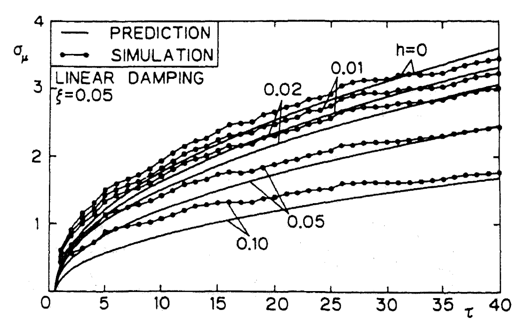

Fig.13 Time change of standard deviation of ductility factor

estimates are satisfactory. σ_λ for the linear damping case only is depicted in Fig.6.

Figures 7 and 8 indicate $\bar{\delta}-\tau$ relations. σ_δ for the linear damping case is shown in Fig.9. The solutions are in good agreement with digital estimates in these figures. $\sigma_\eta-\tau$ relations for the linear damping case are shown in Fig.10. Figures 11 and 12 display $\bar{\mu}-\tau$ relations. σ_μ for the linear damping case is shown in Fig.13. In any cases the degrees of agreement between the both estimates are acceptable from the practical point of view.

It is found from the figures obtained that responses are affected greatly by the magnitude of the damping ratio, whereas slightly by the types of viscous damping in most cases.

8. CONCLUSIONS

The nonlinear random response of the single-degree-of-freedom system having the elasto-plastic hysteretic restoring force together with the viscous damping is presented. The Gaussian white noise is assumed as the ground acceleration. Two types of damping where the coefficient of viscousity is taken proportional to the elastic stiffness and to the instantaneous stiffness are dealt with. The approximate solutions for the expectation and the variance of displacement, velocity, ductility factor, cumulative ductility factor and damping energy are derived on the basis of theoretical investigation. The solutions are compared and well agree with the digital estimates in most cases. The responses are, in general, affected greatly by the magnitude of the damping ratio, whereas slightly by the type of viscous damping.

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