Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5

Dynamic analysis of a strain softening bar

Pankaj & Suraj Parkash
Department of Earthquake Engineering, University of Roorkee, India

ABSTRACT: In aseismic design it often becomes necessary to permit a certain degree of cracking while designing structures to be able to sustain the Maximum Credible Earthquake. This need for post-cracking dynamic analysis has been dealt with by using the concept of strain softening. Numerical solutions using strain softening have been plagued by many problems. This paper highlights attempts by the authors to tackle some of them. An elastoplastic strain softening bar is subjected to dynamic forces in the form of prescribed end velocities. The equations of motion are directly integrated using predictor corrector form of Newmark scheme. A total residual strategy wherein iterations are always related to the converged solution at the end of the previous time step is used to prevent spurious yielding, along with the concept of constant fracture energy release rate to obtain mesh objectivity. The results show a fairly good, localised, high strain region. However, pure elastic unloading adjoining the localised region is delayed.

1 INTRODUCTION

With the continuing construction of structures of importance in the seismic environment it is becoming increasingly important to evolve sophisticated analytical techniques so that these structures can be designed with the required confidence. While designing structures to be able to sustain the Maximum Credible Earthquake, during there lifetime, it becomes imperative, due to economic considerations, to permit some structural components to undergo cracking to a certain extent. This necessitates the need for post-cracking dynamic analysis. The process of progressive failure or damage has been represented with some success in static analysis with the aid of the concept of strain softening (Willam *et al.* 1984; Ottosen 1986; Bazant and Chang 1987; Pankaj 1990; Bicanic and Pankaj 1990a,b; Pankaj and Bicanic 1991; *etc.*), coupled with the smeared crack analysis. Strain softening represents the decline of stress at increasing strain. In the past it has been often argued that softening is unacceptable in analysis as it is not a material property. However, now it is well recognised that softening can be used to provide expedient macroscopic models (Read and Hegemier 1984) in which it is represented in the form of non-local laws (Willam *et al.* 1984; Ottosen 1986; Bazant and Chang 1987). Analytically it has been shown that the length of strain softening region localises into a point, for one dimensional (1-D) problems and some closed form solutions to simple problems have been obtained (Bazant and Belytschko 1985; Belytschko *et al.* 1987; Pankaj 1990). However, numerical solutions using finite element method have been plagued by problems of mesh sensitivity (different solutions being obtained for different meshes), load step sensitivity (Pankaj 1990) and by the inability of capturing the localisation phenomenon. This paper highlights some attempts by the authors in the above direction for a simple dynamic problem.

2 MESH SENSITIVITY

It has been widely agreed that objective, *i.e.* finite element mesh insensitive predictions of localised failure can be obtained if the local material softening law is made mesh dependent, on the basis of constant fracture energy release rate (Hillerborg *et al.* 1976; Bazant and Oh 1983; Willam *et al.* 1985).

For one dimensional problems, however, it is possible to evolve the mesh dependent (length dependent for 1-D) softening modulus without resorting to fracture energy (Pankaj 1990).

Consider a bar of length L in uniaxial tension caused by prescribed end displacement. If this bar is discretized as a single element and it is assumed that it undergoes strain softening, which is confined to a small zone but can be represented in the form of a bilinear stress-strain curve with slope E and E_{s1}, thus asserting that the entire bar is an equivalent continuum element, the displacement of the bar d is

$$d = \epsilon L = \left(\frac{\sigma_f}{E} + \frac{\sigma_f - \sigma}{E_{s1}}\right) L \qquad (1)$$

where σ and σ_f are the stress in the bar and the ultimate stress respectively. If the same bar is discretized into n elements equal in length, but softening is confined to a single element, $n-1$ elements will unload after achieving peak stength while one element will continue to soften. Let the softening slope of this single element be E_{sn}. If the stress in the bar is again σ then the displacement d can be written as

$$d = \left(\frac{n-1}{n}\right)\frac{\sigma}{E}L + \left(\frac{\sigma_f}{E} + \frac{\sigma_f - \sigma}{E_{sn}}\right)\frac{L}{n} \qquad (2)$$

Equating equations 1 and 2 the softening modulus E_{sn} for the n bar discretization, that gives the same stress response as a single bar, can be obtained as a function of E and E_{s1} as

$$E_{sn} = \frac{EE_{s1}}{(n-1)E_{s1} + nE} \qquad (3)$$

Thus without resorting to fracture energy it can be seen that softening modulus cannot be treated as a local material property but has to vary with the mesh discretizations. If the concept of invariant fracture energy release rate is used with the composite damage model (Willam *et al.* 1985) the same result can be obtained (Pankaj 1990).

In plasticity it is more convenient to use the modulus H which is defined as

$$H = \frac{d\sigma}{d\epsilon_p} = \frac{d\sigma}{d\epsilon - d\epsilon_e} = \frac{E_s}{1 - E_s/E} \qquad (4)$$

where $d\epsilon_e$ and $d\epsilon_p$ represent elastic and plastic strain increments respectively. From this the relation between H_1 and H_n (modulii for 1 and n bar discretizations respectively) can easily be derived as

$$H_n = \frac{H_1}{n} \qquad (5)$$

3 LOAD STEP SENSITIVITY

The load sensitivity issue arises when traditional incremental elastoplasticity algorithms are employed. For a simple uniaxial tension test, true localisation solution will involve only one element having entered the post peak range, whereas all other elements will elastically unload. However, for a large displacement, the elastic predictor stresses in more than one (possibly all) elements will exceed tensile strength level. Two iteration strategies have been used to achieve convergence — (a) the subincremental residual strategy and (b) the total residual strategy. These strategies, whose nomenclature as above is due to the first author (1990), can be briefly described as follows.

In subincremental residual strategy the residual (out of balance) forces are calculated with reference to the stress state at the end of every iteration subincrement. Thus plastic strains are accumulated as the stresses are successively being reduced to the current strength limit. Such a strategy in the case of a large a dispacement increment leads to the accumulation of permanent plastic strains in zones that in reality never experience inelastic behaviour. On the other hand, in total residual strategy, the residual forces during iterations are always related to the converged solution at the end of the previous load step and iterations are not treated as subincrements. In this strategy the algorithm itself will "push back", into the elastic zone, the stress points that had "temporarily" violated the tensile strength limit, during computational elastic predictor phase, without accumulation of permanent strains.

4 CAPTURING LOCALISATION

A finite element analysis should be capable of capturing localisation or cracking, which in the limit represents a discontinuity as precisely as possible. It has been suggested (Bazant 1986) that a region that softens cannot be finite and localisation must correspond to infinite strains caused by discontinuous displacements, while the entire domain outside the discontinuity remains elastic. Consequently the best finite element solution is the one which is most localised. However, in a two/three dimensional context, a fine mesh does not guarantee fine localisation bands (Pankaj 1990). It has been argued that meshes need to be designed in a manner, that they are capable of reproducing narrow bands of high inelastic deformation, else only diffused localisation bands may be obtained. For one dimensional problems mesh design in the manner discussed by Bicanic and Pankaj (1990b) is not required. However, difficulties in capturing localisation, in numerical analysis, are still encountered when (a) unequal elements are used (Pankaj 1990); (b) the problem is a bifurcation problem (de Borst 1986) or (c) in case of dynamic problems (Pramono 1988). We shall discuss a simple dynamic problem in the following section and highlight some of the difficulties encountered.

5 A DYNAMIC PROBLEM

Strain softening continues to be a highly researched topic in static as well as dynamic analysis. Some insight on softening and localisation analysis is now available for static analysis. However, limited analytical (Bazant and Belytschko 1985; Belytschko *et al.* 1987), numerical (Pramono 1988; Parkash 1991) and experimental (Gran and Seaman 1988) results are available for dynamic problems. Here a simple wave propagation problem whose exact solution is available is numerically analysed using the predictor corrector form of Newmark's direct integration scheme (Owen and Hinton 1980).

5.1 Problem formulation and exact solution

Consider a bar of length $2L$, with a unit cross-section its mass per unit length is ρ. Let the bar be loaded by forcing both ends to move simultaneously outward, with constant opposing velocity of magnitude c. The boundary conditions are (for $t \geq 0$)

$$u = u(x,t) = \begin{cases} -L & \text{for } u = -ct \\ L & \text{for } u = ct \end{cases}$$

where, $u(x,t)$ is the displacement at x at time t. Initially, at $t = 0$, the bar is undeformed and at rest, *i.e.* $u = \dot{u} = 0$. Due to symmetry this problem is equivalent to a bar fixed at $x = 0$.

The material of the bar has stress strain relationship, which exhibits elastic behaviour with Young's modulus E upto peak stress σ_f, followed by a strain softening curve $F(\epsilon)$, a positive monotonic continuous function which has a negative slope. $F'(\epsilon)$ but, otherwise, an arbitrary shape and which attains zero stress either at some finite strain or asymptotically at infinite strain.

The unloading ($\dot{\epsilon} < 0$) and reloading ($\dot{\epsilon} \geq 0$), upto the last previous maximum strain, is elastic with modulus E and, if the strain increases beyond this maxium the (virgin) strain-softening diagram is followed.

Suppose first that stresses exceeding σ_f (or equivalently strains exceeding ϵ_f) are never produced, *i.e.*, the bar remains linearly elastic. The differential equation of motion is hyperbolic and reads

$$v^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \tag{6}$$

where $v = \sqrt{E/\rho}$ is the wave speed.

For the given boundary and initial conditions, the solution, for $t \leq \frac{2L}{v}$, is (Bazant and Belytschko 1985)

$$u = -c\langle t - \frac{x+L}{v}\rangle + c\langle t + \frac{x-L}{v}\rangle \tag{7}$$

where the Mc Cauley brackets $\langle\cdot\rangle$ extract only the positive values. The strain is

$$\epsilon = \frac{\partial u}{\partial x} = \frac{c}{v}\left[H(t - \frac{x+L}{v} + H(t + \frac{x-L}{v})\right] \tag{8}$$

where H denotes Heaviside step function.

The strain consists of two tensile step waves of magnitude c/v, emanating from the ends of the bar and moving towards the center. After the waves meet at the midpoint, the strain is doubled. The stress $\sigma = E\epsilon$.

Obviously, if $c/v \leq \epsilon_f/2$, then assumption of elastic behaviour holds for $t \leq 2L/v$. If $\epsilon_f/2 < c/v \leq \epsilon_f$, the previous solution holds for $t < L/v$ and the midpoint cross section (x = 0) enters the strain softening regime at $t = L/v$, *i.e.* when the wave fronts meet at the midpoint. For such a case Bazant and Belytschko (1985) derived an exact solution. The complete solution according to them for $0 \leq t \leq 2L/v$ and $x < 0$ is

$$u = -c\langle t - \frac{x+L}{v}\rangle - c\langle t - \frac{L-x}{v}\rangle \tag{9}$$

and

$$\epsilon = \frac{c}{v}\left[H(t - \frac{x+L}{v} - H(t - \frac{L-x}{v})\right] \tag{10}$$

For $x > 0$, a symmetric solution applies, while for

$$x \longrightarrow 0^+, \quad u = -2c\langle t - L/v\rangle$$
$$x \longrightarrow 0^-, \quad u = 2c\langle t - L/v\rangle$$

The superscripts $-$ and $+$ indicate points on the left and right of the interface. So, after time $t_1 = L/v$, the displacements develop a discontinuity at $x = 0$, with a jump of magnitude $4c\langle t - L/v\rangle$. Therefore, strain near $x = 0$, *i.e.* at centre of bar is $\epsilon = 4c\langle t - L/v\rangle\delta(x)$, in which $\delta(x)$ is the Dirac's delta function.

The complete strain field for $x \leq 0$ and $0 \leq t \leq 2L/v$ is,

$$\epsilon = \frac{c}{v}\left[H(t - \frac{x+L}{v} - H(t - \frac{L-x}{v}) + 4\langle vt - L\rangle\delta(x)\right] \tag{11}$$

5.2 Numerical solution

Numerical analysis of a bar of unit area subjected to constant velocity applied at both ends in the outward direction was carried out. The bar was assumed to behave elastically ($E = 2 \times 10^5\ N/cm^2$) upto a critical stess ($\sigma_f = 1.5 \times 10^6\ N/cm^2$) after which it would undergo strain softening with a constant softening modulus.

Two different discretizations were employed for numerical analysis. In the first the bar was divided into 9 elements of equal length and in the other into 21 equal elements. The concept of constant fracture energy release rate for a 1-D problem implies that the softening modulus H be proportional to element size l, *i.e.*

$$H \propto l \tag{12}$$

So, if the element lengths are l_1 and l_2 the softening modulii are related as

$$\frac{H_1}{H_2} = \frac{l_1}{l_2} \tag{13}$$

In other words, for the problem under consideration if the softening modulus $H = -20000\ N/cm^2$ for 9 element discretization, the corresponding softening modulus for 21 element discretization is $-8571.43\ N/cm^2$. In addition to the above corresponding values of softening modulii another corresponding set in which $H = -46666.67\ N/cm^2$ for 9 element dicretization and $H = -20000\ N/cm^2$ for 21 element discretization was adopted for computation. The total residual strategy was adopted. Damping was taken as zero and a diagonal lumped mass matrix was used. The Newmark's parameters, $\tau = 0.5$ and $\delta = 0.25$ were adopted. A time step $\Delta t = 0.1\ sec$ was used.

6 DISCUSSION OF RESULTS

The displacement, total strain and stresses obtained from numerical analysis at different times have been plotted in Figures 1 to 4. Three instants on the time axes were chosen for the plots — 3 *sec* at which the opposite stress waves have not crossed each other; 6.5 *sec* and 7.5 *sec* when the stress waves have already encountered each other and the elastic stress limit has been exceeded.

At 3 *sec* the response remains elastic (Figures 1 and 2). It can be seen that the displacement is more or less as predicted by the closed form solution. The stress and strain waves of approximately constant magnitude progress towards the centre of the bar. Clearly the magnitude of the softening modulus has no effect on the response at 3 *sec*. Also 21 elements discretization produces a response that is closer to the exact solution as compared to the 9 elements discretization.

At 6.5 and 7.5 *sec* the stress waves have already encountered each other. The exact solution predicts that once the stress waves encounter each other the strain in an infinitesimal band at the center would shoot up (delta function response) and a zero magnitude stress wave would travel outward exhibiting unloading. Thereby the localisation of strain would be confined to the centre. In this study it was seen that the post encounter period of the stress waves is accompanied by a steep rise in strains in the central element (Figures 1 to 4).

However, the true localisation solution, wherein elements adjacent to the central element immediately start unloading and plastic strains remain confined to the central element was not achieved. Unloading does start (as has been indicated by symbol U in the above mentioned Figures 1 to 4) in elements adjacent to the central element but this happens subsequent to these elements entering the post-peak strain levels. Strictly, mesh sensitivity issues cannot be discussed as true localisation solution was not achieved, however, it can be seen that

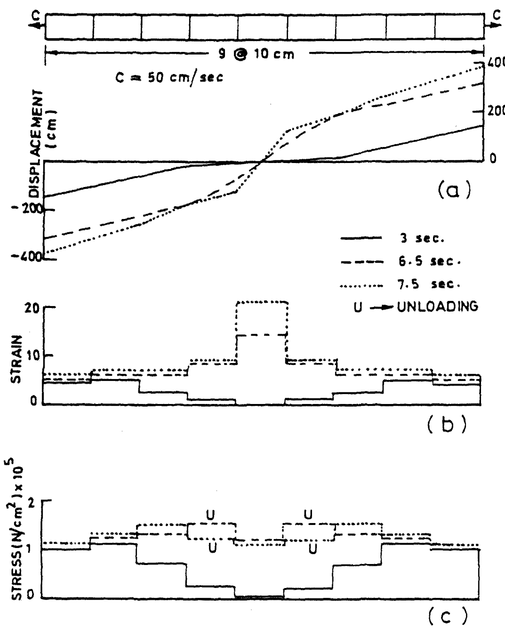

Figure 1: (a) Displacement, (b) strain and (c) stress configurations for 9 element discretization with $H = -20000 N/cm^2$

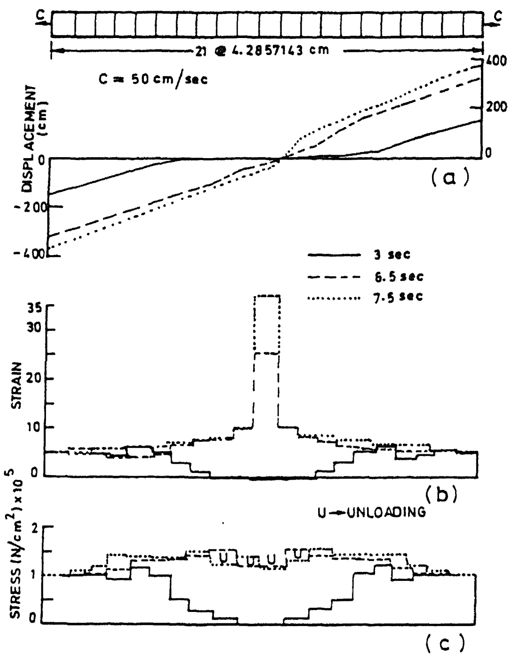

Figure 2: (a) Displacement, (b) strain and (c) stress configurations for 21 element discretization with $H = -8571.4 N/cm^2$

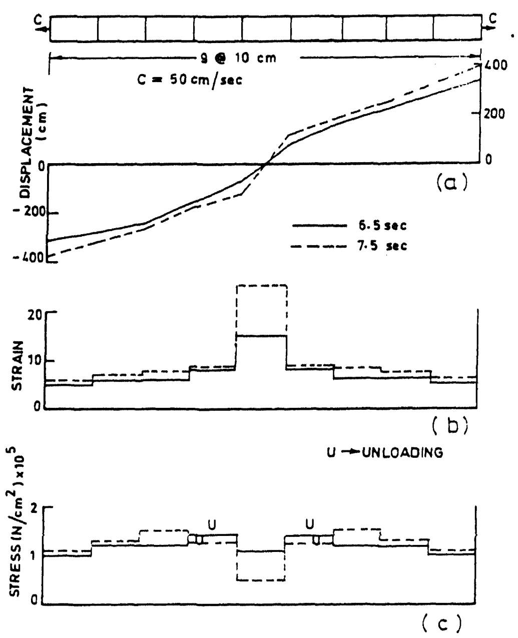

Figure 3: (a) Displacement, (b) strain and (c) stress configurations for 9 element discretization with $H = -46666.7 N/cm^2$

Figure 1 compares well with Figure 2, as far as stress and displacement response is concerned. As can be expected with finer discretization, the strain in the central element is much higher (Figure 2). Similar observations can be made by comparing Figures 3 and 4.

Bazant and Belytschko (1985) showed that the softening modulus has no effect on the postencounter session of the waves. However, numerical analysis shows that higher degree of localisation in the form of a steep strain rise in the centre and quick unloading are achieved when a steeper softening modulus is employed (Figures 3 and 4).

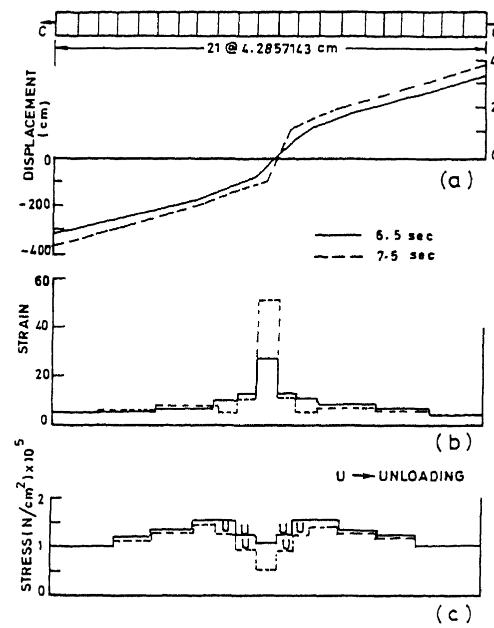

Figure 4: (a) Displacement, (b) strain and (c) stress configurations for 21 element discretization with $H = -20000 N/cm^2$

7 CONCLUSIONS

It can be said that a good localisation is achieved in the form of a strain jump in the central element. The numerical results for the discretized problem are similar to those obtained earlier using explicit integration and viscoplasticity (Pramono 1988). They, however, do not match the results of the continuum problem. First obvious reason is that a discretized problem can never truly represent a continuum problem. Secondly, unlike the continuum case in which a strain jump takes place instantly when the two waves encounter each other, in the discretized case a strain jump is only possible when the two opposing waves have crossed the central element.

Another positive aspect of the numerical analysis, in addition to obtaining high strains in the centre, is that elements near the central element show unloading, though it is delayed. Moreover, improved solutions are obtained with the finer discretization and steeper softening slope.

REFERENCES

Bazant, Z.P. 1986. Mechanics of distributed cracking. *Appl. Mech. Rev.*, 39:675–705.

Bazant, Z.P. and B.H. Oh 1983. Crack band theory for fracture of concrete. *Materials and Structures, RILEM*, 16:155–177.

Bazant, Z.P. and T. Belytschko 1985. Wave propagation in a strain softening bar: Exact solution. *J. Eng. Mech., ASCE*, 111:381–389.

Bazant, Z.P. and T-P. Chang 1987. Nonlocal finite element analysis of strain softening solids. *J. Eng. Mech., ASCE*, 113:89–105.

Belytschko, T., Xiao-Jun Wang, Z.P. Bazant and Y. Hyun 1987. Transient solutions for one dimensional problems with strain softening. *J. Appl. Mech., ASME*, 54:513-518.

Bicanic, N. and Pankaj 1990a. Some computational aspects of tensile strain localization modelling in concrete. *Engineering Fracture Mechanics*, 35: 697-707.

Bicanic, N. and Pankaj 1990b. On mesh design for concrete strain softening analysis. In *Computer Aided Analysis and Design of Concrete Structures*, (N. Bicanic and H. Mang eds.), Vol. 2, Pineridge Press, 959–974.

de Borst, R. 1986. *Non-linear analysis of frictional materials.* Dissertation, Delft University of Technology.

Gran, J.K. and L. Seaman 1988. Strain softening calculations for concrete in dynamic uniaxial tension. *J. Eng. Mech., ASCE*, 114:1911-1928.

Hillerborg, A., M. Modeer and P.E. Petersson 1976. Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cement and Concrete Research*, 6:773–782.

Ottosen, N.S. 1986. Thermodynamic consequences of strain softening in tension. *J. Eng. Mech., ASCE*, 112:1152–1164.

Owen, D.R.J. and E. Hinton 1980. *Finite elements in plasticity: Theory and practice.* Pineridge Press, Swansea.

Pankaj 1990. *Finite element analysis in strain softening and localisation problems.* Ph.D. Thesis, Dept. of Civil Eng. University of Wales, Swansea.

Pankaj and N. Bicanic 1991. Mesh sensitivity issues in strain softening and localisation analyses. Proc. *Asian Pacific Conference on Computational Mechanics*, Hong Kong: Balkema.

Parkash, S. 1991. *On some numerical problems in the analysis of a strain softening bar.* ME Thesis, Dept. of Earthquake Eng., University of Roorkee, Roorkee, India.

Pramono, E. 1988. *Numerical simulation of distributed and localized failure in concrete.* Ph.D. Thesis, Univ. of Colorado, Boulder.

Read, H.E. and G.A. Hegemier 1984. Strain softening of rock, soil and concrete – A review article. *Mech. Mat.*, 3:271–294.

Willam, K.J., N. Bicanic and S. Sture 1984. Constitutive and computational aspects of strain softening and localisation in solids. *ASME-WAM '84 Symposium on constitutive equations: Micro, Macro and Computational Aspects*, New Orleans, (K. Willam ed.), ASME, Vol. 900274, New York, 233–252.

Willam, K.J., N. Bicanic, E. Pramono and S. Sture 1984. Composite damage model for strain computations in concrete. Proc. *Int. Conf. on Fracture Mechanics of Concrete*, Lausanne.