Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5

Dynamic-stiffness matrix of unbounded soil by finite-element cloning

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ABSTRACT: To calculate the dynamic-stiffness matrix of the unbounded soil, the multi-cell cloning algorithm can be used. The basic cloning equation is formulated for each cell, supplemented by a smooth interpolation function of the dynamic-stiffness matrix. The resulting nonlinear equations are solved iteratively, whereby in each iteration for each cell a quadratic eigenvalue problem similar to standard cloning is processed. 2- and 3-cell cloning lead to very accurate results for all frequencies for a vast range of practical problems (homogeneous and inhomogeneous wedges and halfplanes with excavations of different shapes).

1 INTRODUCTION

To analyse dynamic soil-structure interaction based on the substructure method, the dynamic-stiffness matrix of the unbounded soil must be determined. As an alternative to the boundary-element method, which applies an analytical solution to incorporate the radiation condition, the cloning algorithm based solely on the finite-element formulation can be used.

In this ingenious cloning concept the essential notion of infinity is captured by stating that adding a finite part to an infinite quantity does not change its value. The fundamental idea of cloning is illustrated in Fig. 1 for the semi-infinite soil taking the embedment into account. Adding the bounded cell of finite

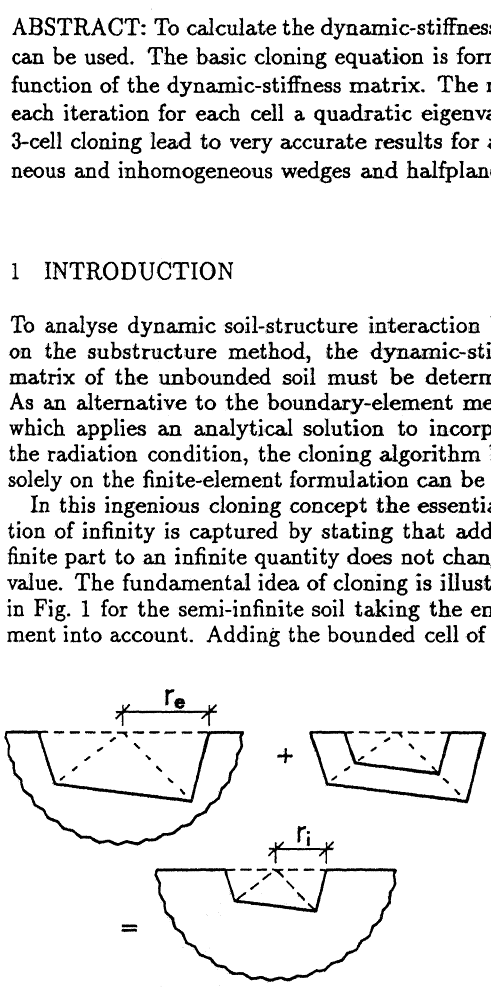

Figure 1: Fundamental concept of cloning algorithm

elements to the unbounded (semi-infinite) domain with the characteristic length r_e results in a similar unbounded domain with length r_i. The concept can be applied to their dynamic-stiffness matrices in the frequency domain by assembling the known dynamic-stiffness matrix of the cell and the unknown matrix of the unbounded soil characterised by the length r_e, which results in the unknown dynamic-stiffness matrix of the unbounded soil with length r_i. As a relationship for the dynamic-stiffness matrices characterised by different lengths exists, the cloning algorithm leads to an expression for the dynamic-stiffness matrix of the unbounded soil as a function of that of the cell, denoted as the basic cloning equation. Potentially, this method is a stand-alone finite-element formulation capable of capturing the radiation condition at infinity without using analytical solutions. Only the conventional static stiffness and mass matrices of the bounded finite-element cell need to be calculated.

In the *standard cloning* algorithm pioneered by Dasgupta (1982) over ten years ago, it is assumed that an average value of the characteristic lengths of the inner and outer boundaries r_i and r_e of the cell can be used in defining the dimensionless frequency of which the dynamic stiffness of the unbounded soil is a function. Or in other words the dynamic stiffnesses of the unbounded soil characterised by the outer and inner boundaries are assumed to be equal. This is not consistent with the derivation of the dynamic stiffness of the cell with the exception of cases where the dimensionless frequencies are the same at the inner and outer boundaries (such as for a soil layer or in the static case). For this special case of constant depth of the layer built-in at its base, Lysmer and Waas (1972) developed essentially the same concept for the dynamic case. The inconsistency present in the general case leads to incorrect results outside the high-frequency range. It is shown in Wolf and Song (1991) that the standard cloning algorithm actually determines the dynamic-stiffness matrix of a different physical system for which the mass density decreases proportionally to the square of the radial coordinate. In particular, an artificial cutoff frequency exists below which no radiation of waves takes place. This is demonstrated in Wolf and Weber (1982). In the same reference, the procedure has been extended to take the variation of the dimensionless frequency from the inner to the outer boundary of the cell into account. This *generalized cloning* method results in ordinary nonlinear first-order differential equations for the dynamic-stiffness matrix with the dimensionless frequency as the independent variable. The system can be solved numerically starting from the value at infinite dimensionless frequency.

Another procedure called *multi-cell cloning* is introduced for the scalar case (i.e. a single dynamic-stiffness coefficient) in Wolf and Song (1991). For n cells, the basic cloning equation can be formulated n times. An additional equation is introduced stating that the $n+1$ dynamic-stiffness coefficients for all boundaries form an $n-1$ degree polynomial of the dimensionless frequency. The standard cloning of Dasgupta (1982) corresponds to one-cell cloning. The multi-cell cloning with 2- or 3-cells as applied to the scalar case leads to a highly accurate dynamic-stiffness coefficient for any specific frequency.

In this paper multi-cell cloning is extended to the matrix case, i.e. the dynamic-stiffness matrix of a foundation with arbitrary geometry of the embedment is determined. In Section 2 the formulation and the solution of the resulting nonlinear equations by iteration are described. The procedure is applied in Section 3 to the calculation of the dynamic-stiffness matrices of several practical cases ranging from the out-of-plane motion of a semi-infinite wedge to the inplane motion of an inhomogeneous halfplane with excavation. A suggestion for further research follows in Section 4.

2 BASIC CLONING EQUATION AND ITS ITERATIVE SOLUTION

For multi-cell cloning, n geometrically similar cells each with its interior and exterior boundaries as shown in Fig. 2a are introduced. This leads to $n+1$ boundaries, each with its own dynamic-stiffness matrix.

For a typical cell j ($j = 1, \ldots, n$) the basic cloning equation is derived as follows (Fig. 2b). The force-displacement relationship of the cell located between the interior and exterior boundaries is written as

$$\left\{ \begin{array}{c} \{P_i\} \\ \{P_e\} \end{array} \right\} = \left[\begin{array}{cc} [S_{ii}] & [S_{ie}] \\ [S_{ei}] & [S_{ee}] \end{array} \right] \left\{ \begin{array}{c} \{u_i\} \\ \{u_e\} \end{array} \right\} \quad (1)$$

where $\{P\}$ and $\{u\}$ are the amplitudes of the nodal forces and displacements. $[S]$ denotes the dynamic-stiffness matrix of the cell composed of finite elements. The corresponding equations for the interior and exterior infinite domains are formulated as

$$\{P_i\} = [S_i^\infty]\{u_i\} \quad (2)$$

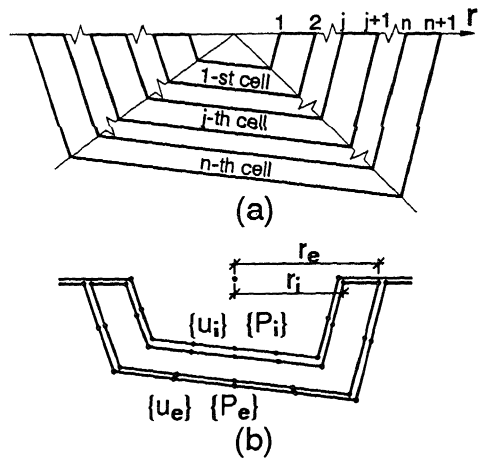

Figure 2. Multi-cell cloning

$$-\{P_e\} = [S_e^\infty]\{u_e\} \quad (3)$$

Eliminating $\{P_i\}$ and $\{P_e\}$ from equations (1), (2) and (3) results in

$$[S_i^\infty] = [S_{ii}] - [S_{ie}]([S_e^\infty] + [S_{ee}])^{-1}[S_{ei}] \quad (4)$$

This basic cloning equation can be formulated for each cell, leading to n sets of equations in $n+1$ sets of unknowns $[S_j^\infty]$ ($j = 1, \ldots, n+1$). To determine the $(n+1)^{th}$ set of equations to supplement equation (4), it is assumed that $[S_j^\infty]$ varies as a smooth interpolation function of the radial distance

$$\sum_{j=1}^{n+1} \frac{l_j}{r_j^{k-2}}[S_j^\infty] = 0 \quad (5)$$

l_j depends on the selected interpolation function and all radial distances of the boundaries. k is the spatial dimension (2 or 3). The following coefficients are used for one-cell cloning (standard cloning)

$$l_1 = -l_2 = 1 \quad (6)$$

for two-cell cloning

$$l_1 = r_3 - r_2 \quad l_2 = r_1 - r_3 \quad l_3 = r_2 - r_1 \quad (7)$$

for three-cell cloning

$$l_1 = \frac{r_1(r_4 - r_2)}{(r_4 - r_1)(r_2 - r_1)} \quad (8a)$$

$$l_2 = \frac{-r_2(r_3 - r_1)}{(r_2 - r_1)(r_3 - r_2)} \quad (8b)$$

$$l_3 = \frac{r_3(r_4 - r_2)}{(r_4 - r_3)(r_3 - r_2)} \quad (8c)$$

$$l_4 = \frac{-r_4(r_3 - r_1)}{(r_4 - r_1)(r_4 - r_3)} \quad (8d)$$

The system of nonlinear equations (4) and (5) describing the multi-cell cloning procedure is solved iteratively as follows. For each cell a quadratic eigenvalue problem is formulated similar to standard cloning. A set of independent $\{u_i\}$ vectors at the interior boundary is combined to form the unit matrix $[I]$. Eliminating $[P_i]$ from equations (1) and (2)

$$[S_i^\infty] = [S_{ii}] + [S_{ie}][u_e] \quad (9)$$

follows, where $[u_e]$ is the corresponding displacement matrix at the exterior boundary. Eliminating $[P_e]$ from equations (1) and (3) results in

$$-[S_e^\infty][u_e] = [S_{ei}] + [S_{ee}][u_e] \quad (10)$$

Multiplying equation (9) by $[u_e]$ and adding it to equation (10) yields

$$[S_{ie}][u_e]^2 + ([S_{ii}] + [S_{ee}] - [S_i^\infty] + [S_e^\infty])[u_e] + [S_{ei}] = 0 \quad (11)$$

When $-[S_i^\infty] + [S_e^\infty]$ is known, this quadratic eigenvalue problem is solved using standard procedures. For $[S_i^\infty] = [S_e^\infty]$, the equation of standard cloning is derived.

To solve equations (9) and (11) formulated for each cell and equation (5) the following iterative procedure is used. In the first iteration $[S_i^\infty] = [S_e^\infty]$ is assumed for each cell. $[u_e]$ follows from equation (11), whereby the same criterion as in standard cloning (Dasgupta 1982) is used to select the eigenvalues, and then $[S_i^\infty]$ is determined from equation (9). This is performed for each cell which leads to $[S_j^\infty]$ for $j = 1, \ldots, n$. Using the latter values, $[S_{n+1}^\infty]$ is calculated using equation (5). For the second iteration $-[S_i^\infty] + [S_e^\infty]$ is determined using the results from the first iteration, which allows the procedure to continue. Convergence is reached when $[S_{n+1}^\infty]$ calculated from equation (5) at the end of an iteration satisfies equation (10) formulated for the n-th cell within an error margin.

3 EXAMPLES

In all calculations the cell width measured in the radial direction is chosen equal to the finite element length in the circumferential direction for the static case and then reduced to guarantee 10 points per wavelength in the dynamic case. Poisson's ratio equals 0.25. The dynamic-stiffness coefficient $S^\infty(a_0)$ as a function of the dimensionless frequency $a_0 = \omega r_0/c_s$ (r_0 = characteristic length as shown in the following figures, c_s = (smallest) shear-wave velocity) is non-dimensionalized as

$$S^\infty(a_0) = K[k(a_0) + ia_0c(a_0)] \quad (12)$$

with the static stiffness K. If K vanishes, the (smallest) shear modulus G is used for the non-dimensionalization.

First, the out-of-plane motion of a wedge with an opening angle $\alpha = 30°$ with a free and a fixed boundary extending to infinity is addressed (Fig. 3). The dynamic-stiffness coefficient corresponding to

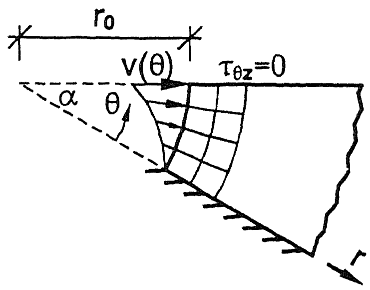

Figure 3: Semi-infinite wedge with prescribed linear displacement and 4 finite elements per cell

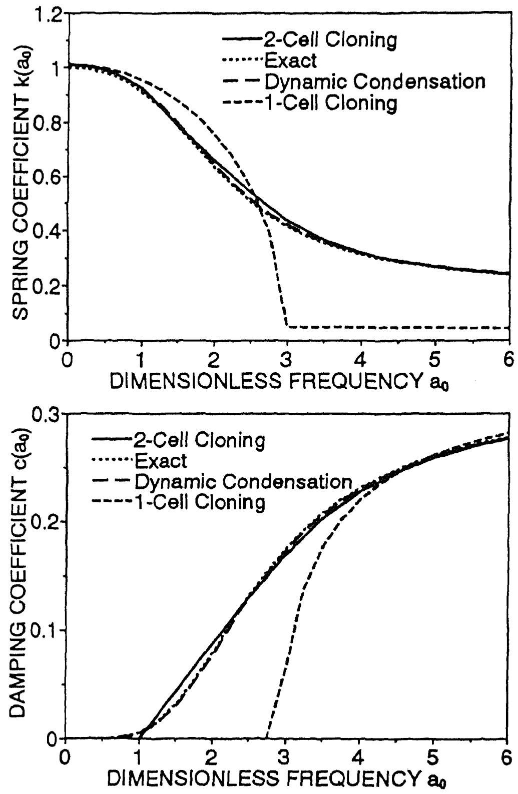

Figure 4: Out-of-plane motion of semi-infinite wedge

the out-of-plane motion $v(\theta)$ prescribed as a linear function in the circumferential direction on the arc is calculated with 4 elements per cell. The 2-cell cloning leads to accurate results (Fig. 4) as is verified when compared with the exact solution which can be found in Wolf and Song (1991). The solution is also presented for dynamic condensation, which consists in applying equation (4) recursively, starting with the initial value at $a_0 = 20$ provided by 2-cell cloning. Throughout the frequency range, this result agrees very well with the exact values. The results of dynamic condensation can thus be used to evaluate the accuracy in cases where no exact solution is avail-

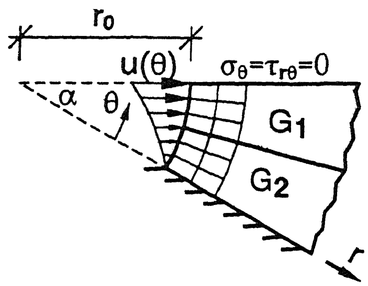

Figure 5: Inhomogeneous semi-infinite wedge with prescribed linear horizontal displacement and 6 finite elements per cell

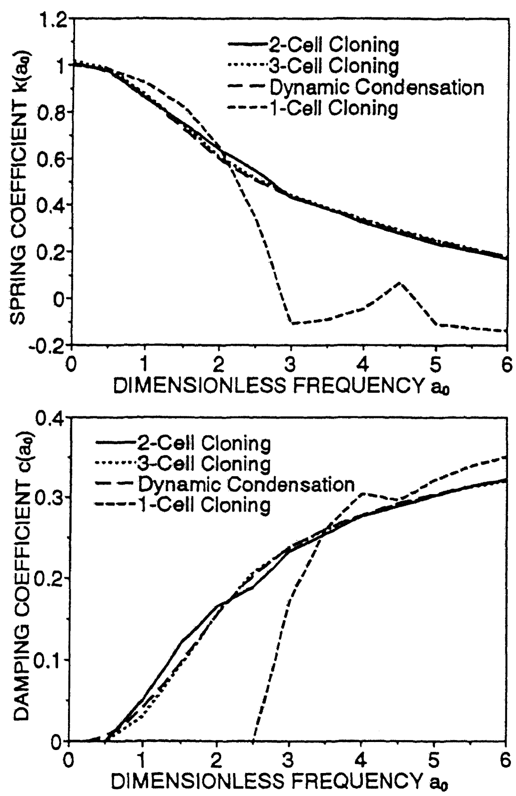

Figure 6: In-plane motion of semi-infinite homogeneous wedge

able. 1-cell cloning (standard cloning) leads to large discrepancies and can not be used.

Second, the in-plane motion of the same wedge is examined. $S^{\infty}(a_0)$ corresponding to a horizontal motion prescribed again as a linear function in the circumferential direction is calculated (Fig. 5), whereby 6 finite elements per cell are chosen. For the homogeneous case ($G_1 = G_2$), 3-cell cloning leads to excellent results for the horizontal dynamic-stiffness coefficient and the solution for 2-cell cloning is highly accurate (Fig. 6). For the inhomogeneous case G_2/G_1=4, 2-

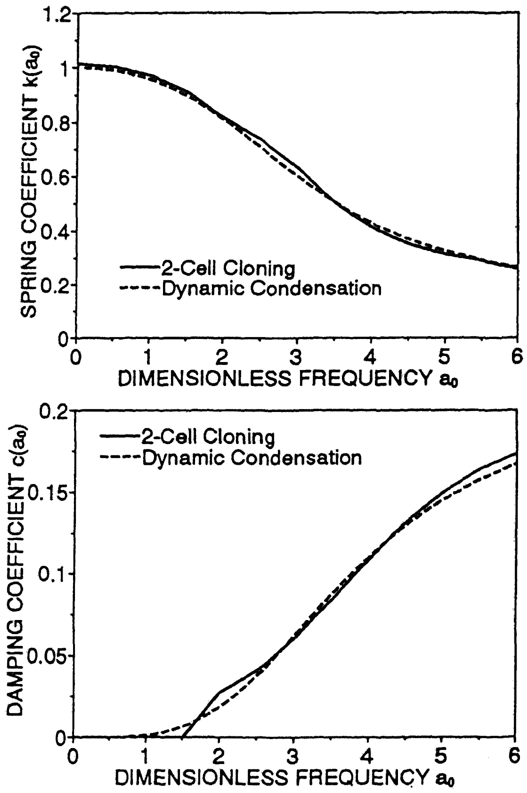

Figure 7: In-plane motion of semi-infinite inhomogeneous wedge

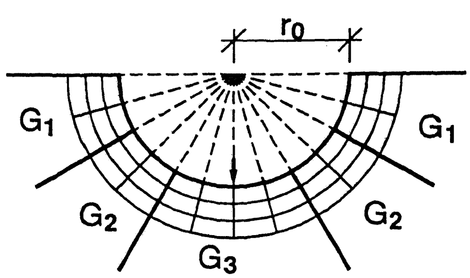

Figure 8: Semi-circular rigid foundation embedded in inhomogeneous halfplane with 12 elements per cell

cell cloning works very well (Fig. 7).

Third, the vertical motion of a semi-circular rigid foundation embedded in a halfplane with 12 elements per cell in the circumferential direction is discussed (Fig. 8). The results of the dynamic-stiffness coefficient for the homogeneous case $G_1 = G_2 = G_3$ (Fig. 9) agree very well with the boundary-element solution specified in Wang and Rajapakse (1991). 3-cell cloning leads to highly accurate results with $G_2/G_1 = 2$, and $G_3/G_1 = 3$ (Fig. 10).

Fourth, a rectangular rigid foundation embedded in a homogeneous halfplane with e/r_0=1 is evaluated by using 20 elements per cell (Fig. 11). Again,

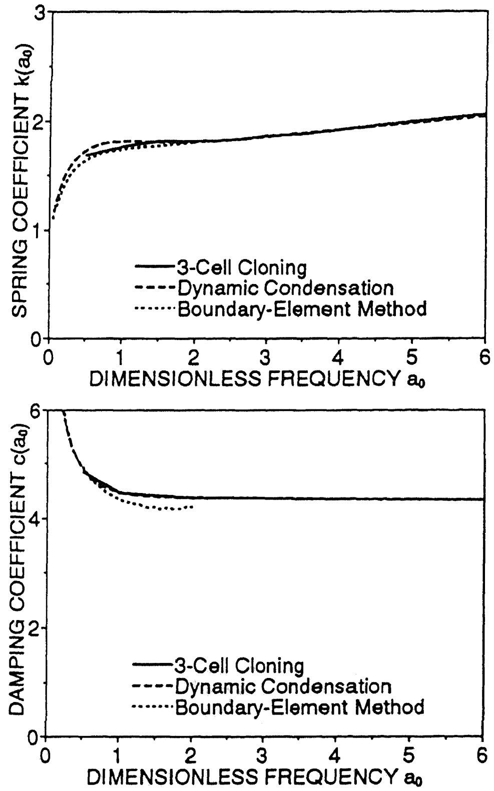

Figure 9: Vertical motion of semi-circular foundation embedded in homogeneous halfplane

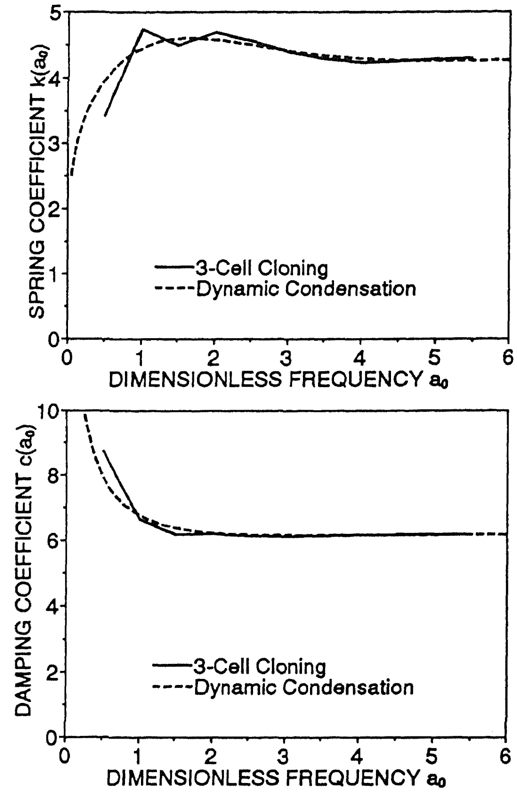

Figure 10: Vertical motion of semi-circular foundation embedded in inhomogeneous halfplane

the dynamic-stiffness coefficients for the horizontal, vertical and rocking motions agree well with the boundary-element solution taken from Wang and Rajapakse (1991) (Fig. 12).

4 SUGGESTION FOR FURTHER RESEARCH

The cloning algorithm is by no means restricted to the analysis of a dynamic system described by one non-dimensional number (which is the dimensionless frequency). By defining further non-dimensional numbers, such as ratios of the dimensions of the foundation and of the site, a dimensional analysis can be performed. The basic cloning equation remains valid. This should allow more general cases to be calculated, such as a foundation of arbitrary shape embedded in a layered halfspace.

ACKNOWLEDGEMENT

The authors wish to express their gratitude to Professor J. Descloux and Mr. H. Débonnaire for in-depth discussions.

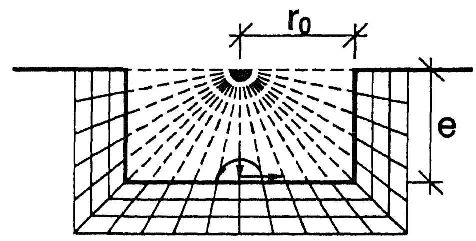

Figure 11: Rectangular rigid foundation embedded in homogeneous halfplane with 20 elements per cell

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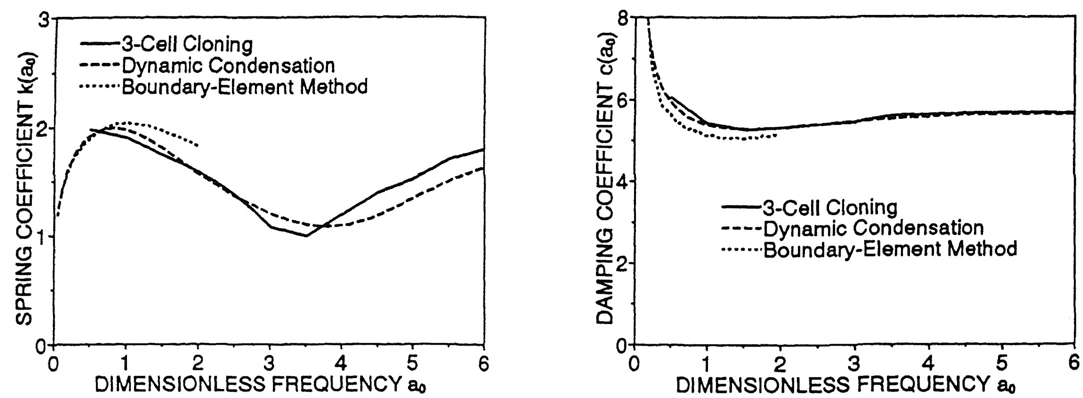
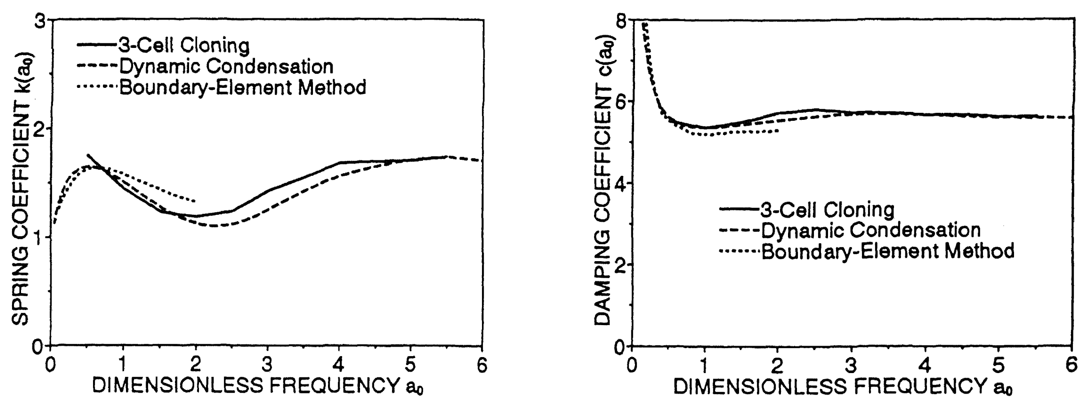
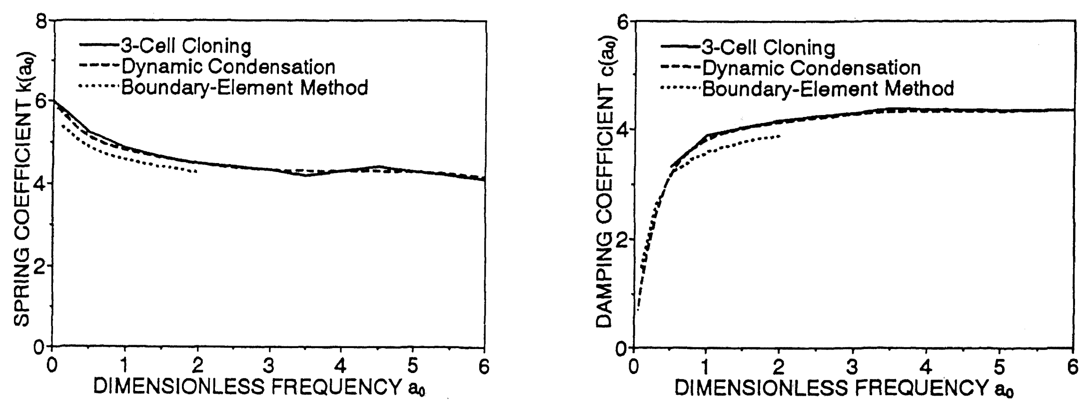


(a) Horizontal



(b) Vertical



(c) Rocking

Figure 12. Rectangular foundation embedded in homogeneous halfplane

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