Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5

Evaluation of Green's function on semi-infinite elastic medium

Toshio Kobayashi
Kajima Technical Research Institute, Kajima Corporation, Japan

Fumio Sasaki
Information Processing Center, Kajima Corporation, Japan

ABSTRACT: In soil structure interaction problem, the Green's function on the surface of semi-infinite elastic medium plays an important role. In evaluating the Green's function, several difficulties occur because it is formulated in the infinite integral form and there is a pole on or near the integration path. This paper describes a method to transform this infinite integral into finite integral based on Cauchy's theorem. Numerical examples obtained by proposed finite integral method are also compared with that obtained by conventional infinite integral method, and the results show fairly good coincidence. Proposed method has advantages on short range [0,1] integral without cutoff error and moderate characteristics of integrands without any difficulties for numerical integration. Proposed method has also an advantage on applicability for both of the cases that the soil material damping is considered and neglected.

1 INTRODUCTION

In order to estimate the dynamic soil-structure interaction effect, it is important to calculate the Green's function for semi-infinite elastic medium considering the free surface condition. In general, this Green's function is formulated in the infinite integral named Fourier-Bessel transform. This integrand has a pole on integration path in case of non-damping, so integral considering residue is inevitable. Also, as the integrand changes steeply in the neighborhood of the pole, several difficulties occur in numerical integration. And due to the infinite integration, large amount of computational time is required. For these reasons, there have been many ideas to calculate the Green's function, e.g. Kobori et.al.(1987).

In this paper, a method which transforms this infinite integral $[0,\infty]$ into a finite integral [0,1], is reported. In the method, an adequate complex function corresponding to the integrand is used and an identical equation which involves the infinite integral is obtained based on Cauchy's theorem. Using this identical equation, the infinite integral of the original solution is replaced by the finite integral.

2 GREEN'S FUNCTION FOR SEMI-INFINITE ELASTIC MEDIUM

The Green's function for point load excitation applied to the surface of semi-infinite elastic medium considering internal damping is examined.

The vertical and radial response displacements of the surface in case of the vertical point load $Pe^{i\omega t}$ applied to the surface ($z = 0$) are given in the cylindrical coordinate system of Fig.1 as follows:

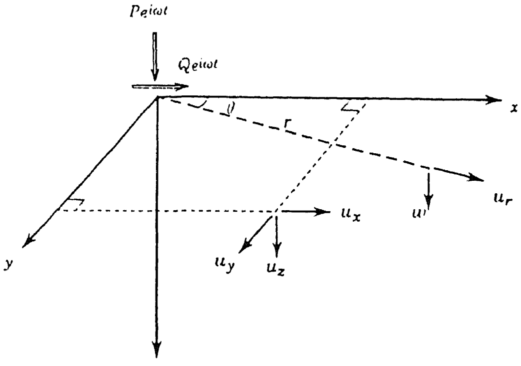

Figure 1. Cylindrical coordinate system

$$w = -\frac{Pe^{i\omega t}}{2\pi G e^{4i\phi}} j \int_0^\infty \frac{\zeta\left(\zeta^2 - \gamma^2 e^{-2i\phi}\right)^{1/2}}{F(\zeta)} J_0(\zeta a)\, d\zeta, \tag{1}$$

$$u_r = \frac{Pe^{i\omega t}}{2\pi G e^{2i\phi}} j \int_0^\infty \frac{E(\zeta)}{F(\zeta)} J_1(\zeta a)\, d\zeta \tag{2}$$

where

$$F(\zeta) = (2\zeta^2 - e^{-2i\phi})^2 - 4\zeta^2(\zeta^2 - \gamma^2 e^{-2i\phi})^{1/2}(\zeta^2 - e^{-2i\phi})^{1/2}, \tag{3}$$

$$E(\zeta) = (2\zeta^2 - e^{-2i\phi})^2 - 2(\zeta^2 - \gamma^2 e^{-2i\phi})^{1/2}(\zeta^2 - e^{-2i\phi})^{1/2}, \tag{4}$$

$\gamma=\sqrt{(1-2\nu)/(2-2\nu)}$, $a = r\omega/Vs$, $j = \omega/Vs$,
Vs : shear wave velocity, ν : Poisson's ratio,
$Ge^{2i\phi}$: Complex shear rigidity, $\phi = \arctan(2h)/2$,
Jn : n-th Bessel function, h : material damping.

Similarly, the horizontal and vertical response displacements of surface in case of the horizontal point load $Qe^{i\omega t}$ applied to the surface (z = 0) are given as follows:

$$u_x = \frac{Qe^{i\omega t}}{4\pi Ge^{2i\phi}} j \int_0^\infty \left[\left\{ \frac{\zeta}{\left(\zeta^2 - e^{-2i\phi}\right)^{1/2}} - \frac{\zeta\left(\zeta^2 - e^{-2i\phi}\right)^{1/2} e^{-2i\phi}}{F(\zeta)} \right\} J_0(\zeta a) + \left\{ \frac{\zeta}{\left(\zeta^2 - e^{-2i\phi}\right)^{1/2}} + \frac{\zeta\left(\zeta^2 - e^{-2i\phi}\right)^{1/2} e^{-2i\phi}}{F(\zeta)} \right\} J_2(\zeta a)\cos 2\theta \right] d\zeta \qquad (5)$$

$$u_y = \frac{Qe^{i\omega t}}{4\pi Ge^{2i\phi}} j \int_0^\infty \left\{ \frac{\zeta}{\left(\zeta^2 - e^{-2i\phi}\right)^{1/2}} + \frac{\zeta\left(\zeta^2 - e^{-2i\phi}\right)^{1/2} e^{-2i\phi}}{F(\zeta)} \right\} J_2(\zeta a)\sin 2\theta d\zeta \qquad (6)$$

$$u_z = \frac{Qe^{i\omega t}}{2\pi Ge^{2i\phi}} j \int_0^\infty \frac{E(\zeta)}{F(\zeta)} J_1(\zeta a) d\zeta \qquad (7)$$

Before performing finite integral transform, these five equations (1), (2) and (5) ~ (7) are deformed as follows:

$$w = \frac{Pe^{i\omega t}}{2\pi Ge^{2i\phi}} \frac{1-\nu}{r}\left[1 - \frac{a}{1-\nu} I_1\right] \qquad (8)$$

$$u_r = \frac{Pe^{i\omega t}}{2\pi Ge^{2i\phi}} j I_4 \qquad (9)$$

$$u_x = \frac{Qe^{i\omega t}}{4\pi Ge^{2i\phi}} \frac{1}{r}\left[(2-\nu) - a \times I_2 + (\nu + a \times I_3)\cos 2\theta\right] \qquad (10)$$

$$u_y = \frac{Qe^{i\omega t}}{4\pi Ge^{2i\phi}} \frac{1}{r}(\nu + a \times I_3)\sin 2\theta \qquad (11)$$

$$u_z = \frac{Qe^{i\omega t}}{2\pi Ge^{2i\phi}} j I_4 \qquad (12)$$

where

$$I_1 = \int_0^\infty \left[\frac{\zeta\left(\zeta^2 - \gamma^2 e^{-2i\phi}\right)^{1/2}}{F(\zeta) e^{2i\phi}} + (1-\nu) \right] J_0(\zeta a) d\zeta \qquad (13)$$

$$I_2 = -\int_0^\infty \left[\frac{\zeta}{\left(\zeta^2 - e^{-2i\phi}\right)^{1/2}} - \frac{\zeta\left(\zeta^2 - e^{-2i\phi}\right)^{1/2}}{F(\zeta) e^{2i\phi}} - (2-\nu) \right] J_0(\zeta a) d\zeta \qquad (14)$$

$$I_3 = \int_0^\infty \left[\frac{\zeta}{\left(\zeta^2 - e^{-2i\phi}\right)^{1/2}} + \frac{\zeta\left(\zeta^2 - e^{-2i\phi}\right)^{1/2}}{F(\zeta) e^{2i\phi}} - \nu \right] J_2(\zeta a) d\zeta \qquad (15)$$

$$I_4 = \int_0^\infty \frac{E(\zeta)}{F(\zeta)} J_1(\zeta a) d\zeta \qquad (16)$$

Using these four Is (s = 1~4), all the displacement components [equations (8) ~ (12)] can be evaluated.

3 CONTOUR INTEGRALS ON COMPLEX HALF PLANE

Consider the following complex integral

$$II_s = \int_c \Phi_s(z) dz \qquad (s = 1\sim4) \qquad (17)$$

where

$$\Phi_1(z) = \left\{ \frac{z\left(z^2 - \gamma^2 e^{-2i\phi}\right)^{1/2}}{F(z) e^{2i\phi}} + C_V \right\} H_0(za) \qquad (18)$$

$$\Phi_2(z) = -\left\{ \frac{z}{\left(z^2 - e^{-2i\phi}\right)^{1/2}} - \frac{z\left(z^2 - e^{-2i\phi}\right)^{1/2}}{F(z) e^{2i\phi}} - {}_HC_1 \right\} H_0(za) \qquad (19)$$

$$\Phi_3(z) = \left\{ \frac{z}{\left(z^2 - e^{-2i\phi}\right)^{1/2}} + \frac{z\left(z^2 - e^{-2i\phi}\right)^{1/2}}{F(z) e^{2i\phi}} - \nu \right\} H_2(za) \qquad (20)$$

$$\Phi_4(z) = \frac{E(z)}{F(z)} H_1^{(1)}(za) \qquad (21)$$

where $C_V = 1 - \nu$, ${}_HC_1 = 2 - \nu$, $H_n^{(m)}(z)$ is m-th kind

Hankel function of order n and $H_n(z)$ is introduced by Kobayashi (1981) as modified Hankel function of order n defined as

$$H_n(z) = \frac{1}{\pi}\int_0^{\pi} \exp\{-i(n\theta - z\sin\theta)\}\, d\theta \tag{22}$$

which has an advantage to Hankel function of having no singularity on the point of z = 0.

Rayleigh pole $\zeta_{d0} = \zeta_0 e^{-i\phi}$ and branch points $e^{-i\phi}$ and $\gamma e^{-i\phi}$ are located on the straight line shown in Fig.2.

Integration path C is a closed curve shown in Fig.3 considering these branch points.

Because multivalued functions $(z^2 - e^{-2i\phi})^{1/2}$ and $(z^2 - \gamma^2 e^{-2i\phi})^{1/2}$ are involved in the integrands, the Riemann surface is used.

Table 1 is the list of the holomorphic branches on the Riemann surface of the multivalued functions. Integrals concerned in each part of the integration path can be expressed using Table 1.

4 RESIDUE THEOREM

Integrands are holomorphic in the interior region of integration path C except the Rayleigh pole (ζ_{d0}). Therefore, the Residue Theorem holds as follows:

$$II_s = 2\pi i \mathrm{Res}\{\Phi_s(z);\zeta_{d0}\} \tag{23}$$

It can be converted to

$$\left(\int_{L_- + L_+} + \int_{L_1+L_2+L_3+L_4} + \int_{C_R}\right)\Phi_s(z)\, dz = 2\pi i \mathrm{Res}\{\Phi_s(z);\zeta_{d0}\} \tag{24}$$

From the definition of I(s) by equations (13) ~ (16),

$$\int_{L_- + L_+} \Phi_s(z)\, dz = 2I_s \tag{25}$$

holds in consideration of

$$H_n(x) + (-1)^n H_n(-x) = 2J_n(x), \tag{26}$$

$$H_n^{(1)}(x) - (-1)^n H_n^{(1)}(-x) = 2J_n(x), \tag{27}$$

where x is real and n is integer.

Furthermore, integrals on half circle (C_R) converge to 0 when radius R tends to infinity.

$$\int_{C_R} \Phi_s(z)\, dz \to 0 \qquad (R \to \infty). \tag{28}$$

As the result, Is (s = 1 ~ 4) are obtained by following finite integrals.

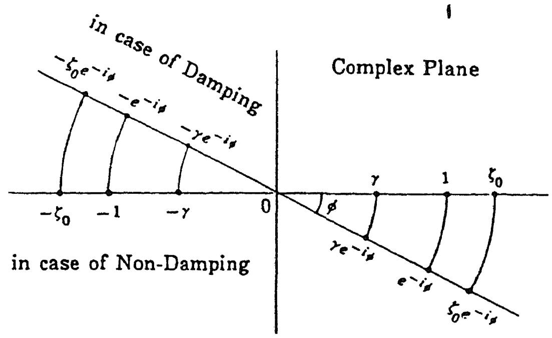

Figure 2. Location of singular points

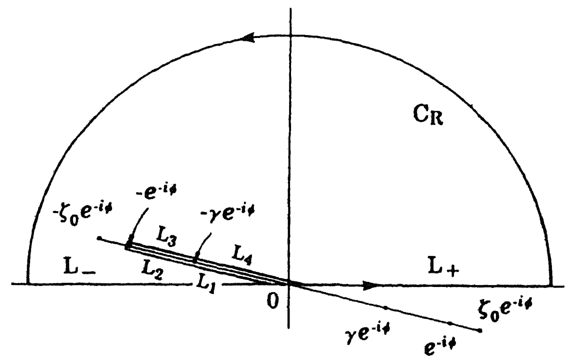

Figure 3. Integration path

Table 1. Holomorphic branches of multivalued functions on the Riemann surface

Path	Domain of z		$(z^2 - \gamma^2 e^{-2i\phi})^{1/2}$	$(z^2 - e^{-2i\phi})^{1/2}$
L_-	$-\zeta < 0$		$-(\zeta^2 - \gamma^2 e^{-2i\phi})^{1/2}$	$-(\zeta^2 - e^{-2i\phi})^{1/2}$
L_1	$z = -re^{-i\phi}$	$0 < r < \gamma$	$-ie^{-i\phi}\sqrt{\gamma^2 - r^2}$	$-ie^{-i\phi}\sqrt{1 - r^2}$
L_2		$\gamma < r < 1$	$-e^{-i\phi}\sqrt{r^2 - \gamma^2}$	
L_3		$1 > r > \gamma$		$ie^{-i\phi}\sqrt{1 - r^2}$
L_4		$\gamma > r > 0$	$ie^{-i\phi}\sqrt{\gamma^2 - r^2}$	
L_+	$\zeta > 0$		$(\zeta^2 - \gamma^2 e^{-2i\phi})^{1/2}$	$(\zeta^2 - e^{-2i\phi})^{1/2}$

$$I_s = \pi i \text{Res}\{\Phi_s ; \zeta_{do}\} - \frac{1}{2}\int_{L_1+L_2+L_3+L_4} \Phi_s(z)\, dz \,. \tag{29}$$

For each Is (s = 1 ~ 4), final expressions are given as follows with reference to Table 1.

$$I_1 = \pi i \text{Res}\{\Phi_1(z) ;\zeta_{d0}\} + ie^{-i\phi}\int_0^\gamma \frac{r\sqrt{\gamma^2 - r^2}}{F_1(r)} H_0(-re^{-i\phi}a)\, dr$$

$$+ ie^{-i\phi}\int_\gamma^1 \frac{4r^3\sqrt{1-r^2}}{G(r)} H_0(-re^{-i\phi}a)\, dr \,, \tag{30}$$

$$I_2 = \pi i \text{Res}\{\Phi_2(z) ;\zeta_{d0}\}$$

$$+ ie^{-i\phi}\int_0^\gamma \frac{r\sqrt{1-r^2}}{F_1(r)} H_0(-re^{-i\phi}a)\, dr$$

$$+ ie^{-i\phi}\int_\gamma^1 \frac{r\sqrt{1-r^2}(2r^2-1)^2}{G(r)} H_0(-re^{-i\phi}a)\, dr$$

$$+ ie^{-i\phi}\frac{\pi}{2}\int_0^1 \sin\frac{\pi}{2}\xi \cdot H_0\left(-\sin\frac{\pi}{2}\xi \cdot e^{-i\phi}a\right) d\xi \,, \tag{31}$$

$$I_3 = \pi i \text{Res}\{\Phi_3(z) ;\zeta_{d0}\}$$

$$+ ie^{-i\phi}\int_0^\gamma \frac{r\sqrt{1-r^2}}{F_1(r)} H_2(-re^{-i\phi}a)\, dr$$

$$+ ie^{-i\phi}\int_\gamma^1 \frac{r\sqrt{1-r^2}(2r^2-1)^2}{G(r)} H_2(-re^{-i\phi}a)\, dr$$

$$- ie^{-i\phi}\frac{\pi}{2}\int_0^1 \sin\frac{\pi}{2}\xi \cdot H_2\left(-\sin\frac{\pi}{2}\xi \cdot e^{-i\phi}a\right) d\xi \,, \tag{32}$$

$$I_4 = \pi i \text{Res}\{\Phi_4(z) ;\zeta_{d0}\}$$

$$+ 2ie^{-i\phi}\int_\gamma^1 \frac{r^2(2r^2-1)\sqrt{1-r^2}\sqrt{r^2-\gamma^2}}{G(r)} H_1^{(1)}(-re^{-i\phi}a)\, dr, \tag{33}$$

where

$$F_1(r) = (2r^2-1)^2 + 4r^2\sqrt{1-r^2}\sqrt{\gamma^2-r^2} \,, \tag{34}$$

$$G(r) = (2r^2-1)^4 + 16r^4(1-r^2)(r^2-\gamma^2) \,. \tag{35}$$

5 NUMERICAL EXAMPLES

5.1 *Integrand*

Figures 4 ~ 9 show the shapes of integrand of transformed finite integral in equations (30) ~ (33) as the case of Poisson's ratio $\nu = 1/3$ and non-dimensional frequency a = 10. The values of the integrand of infinite integral in equations (1),(2) and (5) ~ (7) change steeply near the Rayleigh pole (Kobori (1987)). On the contrary, the shapes of integrand in equations (30) ~ (33) are very smooth as shown in Figures 4 ~ 9. So numerical integration can be carried out with high accuracy without any difficulties.

In calculating $H_n(z)$ and $H_n^{(m)}(z)$, the expansion of infinite series are used.

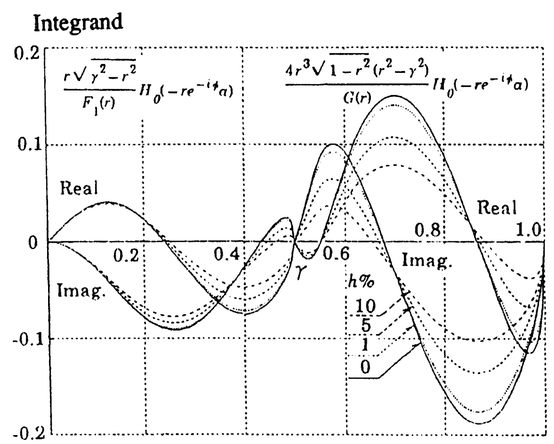

Figure 4. Integrand of vertical fundamental function ($\nu = 1/3$, a = 10)

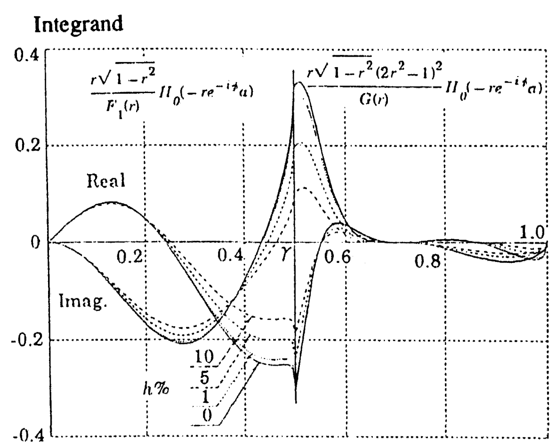

Figure 5. Integrand of horizontal fundamental function of axi-symmetric component ($\nu = 1/3$, a = 10)

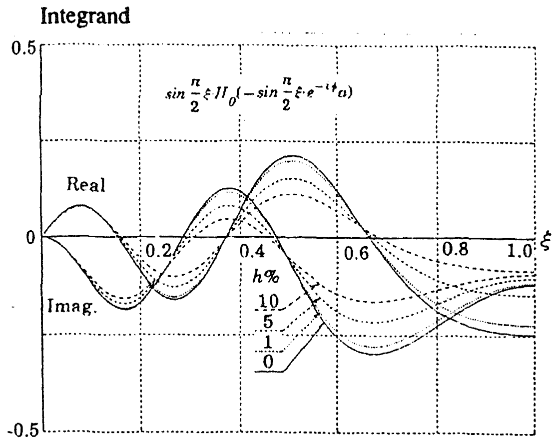

Figure 6. Integrand of horizontal fundamental function of axi-symmetric component (ν = 1/3, a = 10)

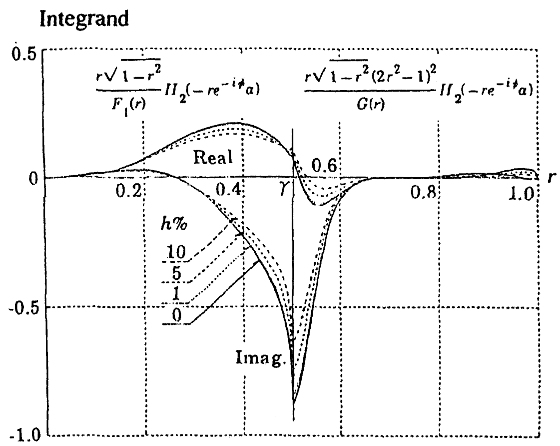

Figure 7. Integrand of horizontal fundamental function of anti-symmetric component (ν = 1/3, a = 10)

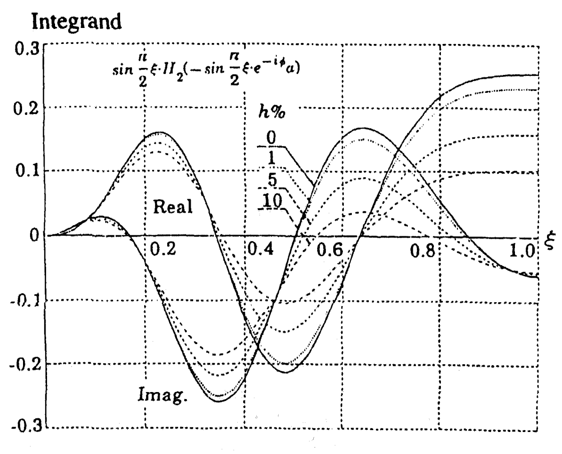

Figure 8. Integrand of horizontal fundamental function of anti-symmetric component (ν = 1/3, a = 10)

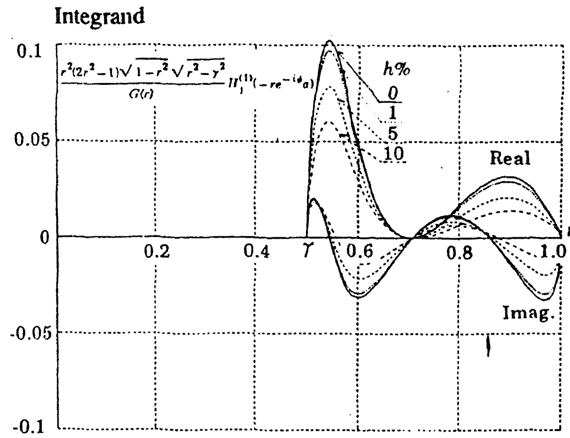

Figure 9. Integrand of orthogonal fundamental function (ν = 1/3, a = 10)

5.2 *Fundamental function*

The Green's functions are represented as

$$w = -\frac{Pe^{i\omega t}}{2\pi G e^{2i\phi}} \frac{1-\nu}{r} \left({}_{v}f_1 + i\,{}_{v}f_2\right), \tag{36}$$

$$u_x = \frac{Qe^{i\omega t}}{4\pi G e^{2i\phi}} \frac{1}{r} \left\{{}_{H}f_1 + i\,{}_{H}f_2 + \left({}_{H}f_3 + i\,{}_{H}f_4\right)\cos 2\theta\right\}, \tag{37}$$

$$u_z = \frac{Qe^{i\omega t}}{2\pi G e^{2i\phi}} \left({}_{0}f_1 + i\,{}_{0}f_2\right), \tag{38}$$

where f_s are called here as fundamental functions.

These fundamental functions are defined by three parameters, which are Poisson's ratio (ν), nondimensional frequency (a), and (ϕ) defined from soil material damping constant h.

With the purpose to confirm the adequacy of the present method, the fundamental functions obtained by present method (solid lines) are compared with those obtained by other method (FFBT method (Ref.[3]), broken lines) *in Figures 10 ~ 13.*

The fundamental functions are calculated in 125 steps from a = 0 to 25 at an interval of Δa = 0.2. Poisson's ratio (ν) is considered to be 1/3. As for soil material damping (h), two cases of 5% and 10% are used

The results show fairly nice coincidence to each other in all components except for slight difference in high frequency range.

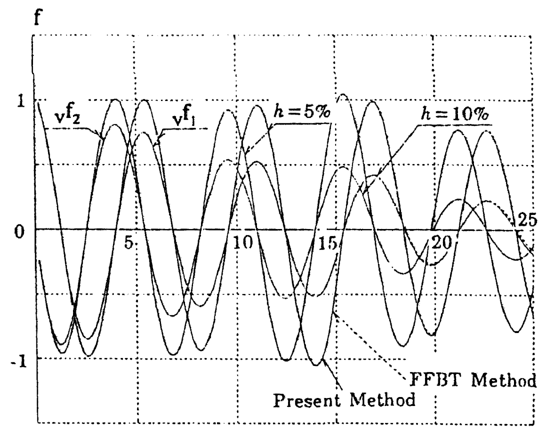

Figure 10. Vertical fundamental function (ν = 1/3)

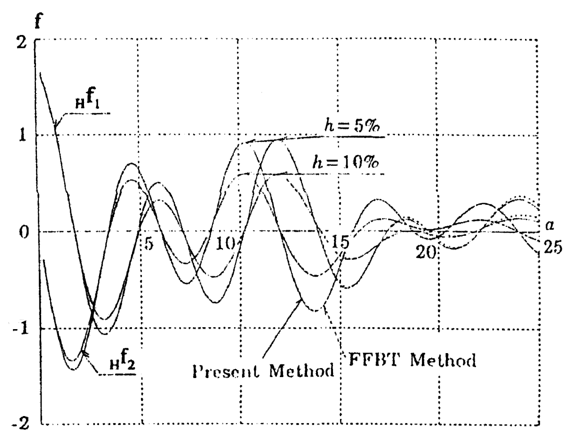

Figure 11. Horizontal fundamental function of axi-symmetric component (ν = 1/3, a = 10)

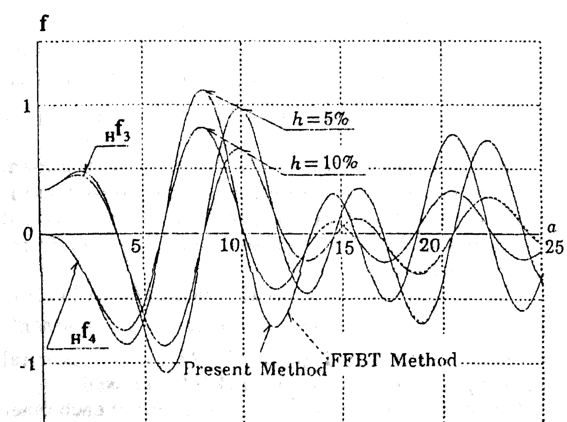

Figure 12. Horizontal fundamental function of anti-symmetric component (ν = 1/3, a = 10)

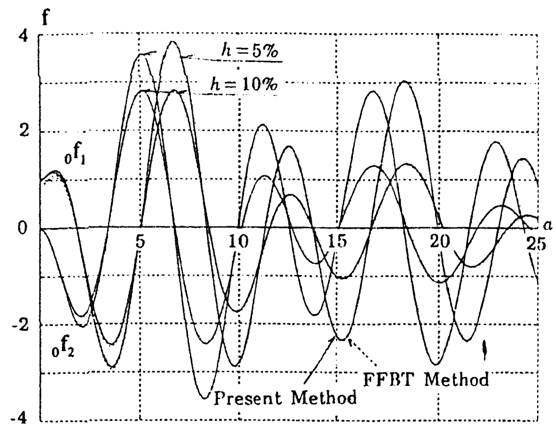

Figure 13. Orthogonal fundamental function (ν = 1/3)

6 CONCLUSIONS

The finite integral transformation of Green's function on the surface of semi-infinite elastic medium has been reported. The adequacy has been investigated with numerical examples.

The advantages of this method are as follows:

1. If the numerical integral procedure is applied directly to the infinite integral form, it becomes impossible to execute it to an infinite range and so must be stopped at a suitable range. If done this way, the cut off error remains, but when the infinite integral is transformed to the finite integral, this cut off error does not appear.
2. Required numerical integral range is very short.
3. In the infinite integral form, there is a pole on or near the integration path. Near the pole, a special procedure is required because the value of the integrand changes so steeply. On the contrary, there is no pole on or near the integration path in the present method, thus, there is no difficulty in the numerical integral process.
4. Present method is applicable for both cases that soil material damping is considered and neglected.

REFERENCES

Kobayashi, T. 1981. Evaluation of Response to Point Load Excitation on Semi-Infinite Elastic Medium (in Japanese). *Transactions of Architectural Institute of Japan*, vol 302, p29 ~ 35.

Kobori, T., K. Miura, T. Moroi, K. Masuda, F. Sasaki, 1987. Fast Fourier Bessel Transforms for Calculating the Green's Function for Semi-Infinite Soil Media. *3rd International Conference on Soil Dynamics and Earthquake Engineering*, Princeton: U.S.A.

Lamb, H. 1904. On the Propagation of Tremors over the Surface of an Elastic Solid. *Philosophical Transactions*, Royal Society: London. A203, p1 ~ 42.