*Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5*

# Closed-form solutions for SRSS response of shear beams

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ABSTRACT : Closed-form solutions are derived for Square Root of the Sum of the Squares (SRSS) seismic analysis of power-law shear beams (commonly used for modeling of soil columns and earthfill dams), when the excitation is defined by a typical regulatory spectrum (constant acceleration up to a given period $T_0$, sloping branch for periods greater thant $T_0$). These solutions are expressed either in exact form (involving Bessel functions) or in approximate form. They should permit to improve simplified methods (mostly based so far on consideration of the first mode alone) for preliminary design of dams, site effects of alluvium layers and design of end-bearing and floating piles.

## 1 INTRODUCTION

The shear beam model is extensively used for seismic analysis of soil columns and earthfill dams. In most cases, analysis follows the modal response spectrum approach, using "Square Root of the Sum of the Squares (SRSS) as the combination rule for modes. This requires the computation of an appropriate number of natural periods and mode shapes, a rather light task for a shear beam model with present-day computers. Nevertheless, simplified methods based upon various approximations of the fundamental mode remain popular for preliminary design purposes.

It has been shown recently, Matsushima (1984), Betbeder-Matibet (1989), that for some particular cases of ground response spectrum, closed-form solutions could be derived for SRSS analysis of a number of shear beam models. The first of these solutions is the formula obtained by Matsushima (1984) for a constant pseudo-velocity spectrum :

$$V(z) = K \times PSV \times [m(z)]^{1/2} \qquad (1)$$

in which $V(z)$ is the shear force at elevation z, $m(z)$ the mass of the part of the beam above this elevation, PSV the pseudo-velocity value which defines the spectrum and K a constant which depends on the mass and stiffness distributions of the beam.

Equation (1) is exact if the effective modal mass is proportional to the square of the period for all the modes of the beam. This modal property, which may be termed Matsushima's assumption, holds true for "power-law shear beams", i.e. shear beams for which mass and stiffness distribution follow power-law in $(H - z)^\alpha$, H being the total height and $\alpha$ an arbitrary constant. For most other cases of shear beams, Matsushima's assumption is only approximate but equation (1) still provides very good results, especially when the beam characteristics vary continuoulsy along its height.

The object of this paper is to provide closed-form solution for power-law shear beams when the ground acceleration response spectrum has the shape of typical regulatory spectra, i.e. constant acceleration $A_0$ up to a given period $T_0$ with a sloping branch $A_0(T_0/T)^\beta$ for $T > T_0$. Such solutions should result in a significant improvement of simplified methods for soil columns and earthfill dams, due to the actual consideration of higher modes and the flexibility of power-law distribution to represent real soil profiles.

A more general theory, including flexion beams as well, and taking into account elastic and fixed base support conditions is given in Betbeder-Matibet (1992), in which the spectrum can be expressed as the square root of an arbitrary polynomial with the square of the period as the independent variable.

## 2 SRSS SOLUTIONS FOR A SHEAR BEAM

For a shear beam with mass density $\rho$, shear modulus G and section S, the equation of motion is :

$$\rho S \left(\frac{\partial^2 u}{\partial t^2} + \ddot{u}_g\right) = \frac{\partial V}{\partial z} \qquad (2)$$

u being the displacement relative to the base, $\ddot{u}_g$ the acceleration of the base and $V = GS\,\frac{\partial u}{\partial z}$ the shear force. The boundary conditions are :

$$(u)_{z=0} = 0 \;;\; (V)_{z=H} = 0 \qquad (3)$$

Noting :

$$\rho o = (\rho)z{=}0 \quad ; \quad Go = (G)z{=}o \; = \; \rho o \; C_o^2 \quad ;$$

$$So = (S)z{=}o$$

$$m(z) = \int_z^H \rho(z')\, S(z')\, dz' \; ; \; m_o = m(0) \; ; \; \omega_o = \frac{\rho_o C_o S_o}{m_o}$$

$$\xi = \frac{m(z)}{m_o} \; ; \; \phi = \frac{\rho \; G \; S^2}{\rho_o \; G_o \; S_o^2}$$

Eqs (2) and (3) can be rewritten as follows :

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \ddot{u}_g = \omega_o^2 \; \dfrac{\partial}{\partial \xi}\left(\phi \dfrac{\partial u}{\partial \xi}\right) \\ (u)_{\xi=1} = 0 \; ; \left(\phi \dfrac{\partial u}{\partial \xi}\right)_{\xi=0} = 0 \end{cases} \tag{4}$$

Natural modes $u_n(\xi)$ corresponding to circular frequencies $\omega_n$, n = 1, 2, ... are defined by :

$$\begin{cases} \dfrac{dV_n}{d\xi} = \Omega_n^2 \, u_n \left(\text{with } V_n = -\phi \dfrac{du_n}{d\xi} \; , \Omega_n = \dfrac{\omega_n}{\omega_o}\right) & (5) \\ (u_n)_{\;\xi=1} = 0 \; ; (V_n)_{\;\xi=0} = 0 & (6) \end{cases}$$

Participation factors $P_n$ and non-dimensional effective modal masses $\mu_n$ are :

$$P_n = \left[\int_o^1 u_n \, d\,\xi\right] \Big/ \left[\int_o^1 u_n^2 \, d\,\xi\right] \tag{7}$$

$$\mu_n = P_n^2 \int_o^1 u_n^2 \; d\,\xi \; \left(\text{with} \sum_n \mu_n = 1\right) \tag{8}$$

When the solution of Eq. (4) is sought through a modal development :

$$u\,(\xi,t) = \sum_n r_n\,(t) \;\; u_n\,(\xi) \tag{9}$$

one has, for shear force V and absolute acceleration a :

$$V\,(\xi,t) = m_o \, \omega_o^2 \sum_n r_n\,(t)\; V_n\,(\xi) \tag{10}$$

$$a\;(\xi,t) = -\;\omega_o^2 \sum_n \Omega_n^2 \, r_n\,(t)\;\; u_n\,(\xi) \tag{11}$$

It stems from the definition of pseudo-acceleration response spectrum $S_a\,(\omega)$ that :

$$\underset{t}{\text{Max}} \; |r_n\,(t)| = \frac{1}{\omega_o^2} \; |P_n| \; S_a \;\; (\omega_n)\,/\,\Omega_n^2 \tag{12}$$

and, using Eqs (9), (10) and (11), the SRSS solutions are found to be :

$$u_m\;(\xi) = \frac{1}{\omega_o^2}\left[\sum_n P_n^2 \, u_n^2\,(\xi)\, S_a^2 \;\; (\omega_n) \;/\; \Omega_n^4\right]^{1/2} \tag{13}$$

$$V_m\;(\xi) = m_o\left[\sum_n P_n^2 \, V_n^2\,(\xi)\, S_a^2 \;\; (\omega_n) \;/\; \Omega_n^4\right]^{1/2} \tag{14}$$

$$a_m\;(\xi) = \left[\sum_n P_n^2 \, u_n^2\,(\xi)\, S_a^2 \;\; (\omega_n)\right]^{1/2} \tag{15}$$

## 3 NATURAL MODES OF POWER-LAW SHEAR BEAMS

A power-law shear beam is defined by :

$$\rho = \rho_o \, ; \, G = G_o\left(1 - \frac{z}{H}\right)^{\alpha G} \; ; \; S = S_o\left(1 - \frac{z}{H}\right)^{\alpha S} \tag{16}$$

$\alpha_G$ , $\alpha_S$ being arbitrary constants ($\alpha_G \geq 0$, $\alpha_S \geq 0$, $\alpha_G < 2$).

For such a beam, the quantities $\xi$ and $\phi$ previously defined are :

$$\xi = \left(1 - \frac{z}{H}\right)^{1+\alpha S} \; ; \; \phi = \xi^\alpha \text{ with } \alpha = \frac{\alpha_G + 2\,\alpha_S}{1+\alpha_S} \tag{17}$$

and the natural modes have the following expression (Dakoulas 1985), Elgamal 1991) :

$$u_n\,(\xi) = \frac{2}{\zeta_n \, J_{\lambda+1}\,(\zeta_n)} \; \xi^{\frac{1-\alpha}{2}} \; J_\lambda \left(\zeta_n \, \xi^{\frac{2-\alpha}{2}}\right) \tag{18}$$

with $\lambda = \dfrac{\alpha - 1}{2 - \alpha}$ and $\zeta_n$ being the zeros of Bessel function $J_\lambda$ ; with this definition all $P_n$ are equal to 1.

The non-dimensional natural circular frequencies $\Omega_n$ are given by :

$$\Omega_n = \left(1 - \frac{\alpha}{2}\right) \zeta_n \tag{19}$$

Interesting particular cases of these power-law shear beams are :

1. $\alpha_G = 0$ , $\alpha_S = 0$ homogeneous soil column

$$\alpha = 0 \, , \lambda = -\frac{1}{2} \, , u_n\,(\xi) = -\,(-1)^n \frac{4}{(2n-1)\,\pi} \cos\left[(2n-1)\,\pi\,\frac{\xi}{2}\right]$$

2. $\alpha_G = \frac{2}{3}$, $\alpha_S = 1$ non homogeneous triangular dam (Gazetas 1980)

$$\alpha = \frac{4}{3} \, , \lambda = \frac{1}{2} \, , u_n\,(\xi) = -\,(-1)^n \frac{2}{n\,\pi\,\xi^{1/3}} \sin\left[(n\,\pi\,\xi^{1/3})\right]$$

For power-law shear beams, the product $\mu_n \, \Omega_n^2$ can be shown to be equal to $2 - \alpha$ for all modes. This is achieved by multiplying both sides of Eq. (5) by $2\,\xi\,\dfrac{du_n}{d\xi}$, thus obtaining

$$\xi^{1-\alpha}\, d\,(V_n^2) + \Omega_n^2 \, \xi \, d\,(u_n^2) = 0$$

Integrating by parts from $\xi = 0$ to $\xi = 1$ and using boundary conditions results in :

$$[V_n^2]_{\xi=1} + (1-\alpha)\int_0^1 V_n\, du_n - \Omega_n^2 \int_0^1 u_n^2\, d\xi = 0$$

which is readily shown to be equivalent to :

$$\Omega_n^2\left[\int_0^1 u_n\, d\xi\right]^2 = (2-\alpha)\int_0^1 u_n^2\, d\xi$$

$$\text{i.e. } \mu_n \Omega_n^2 = 2-\alpha \qquad (20)$$

Matsushima's assumption is valid for these beams.

## 4 SRSS SOLUTIONS FOR POWER-LAW SHEAR BEAMS IN CASE OF A CONSTANT ACCELERATION SPECTRUM

Let us consider the following functions :

$$\begin{cases} w_{1,0}(\xi,\xi') = 1 & 0 \le \xi < \xi' \\ w_{1,0}(\xi,\xi') = 0 & \xi' < \xi \le 1 \end{cases} \qquad (21)$$

$$\begin{cases} w_{2,0}(\xi,\xi') = \frac{1}{1-\alpha}\left[1 - \xi'^{\,1-\alpha}\right] & 0 \le \xi \le \xi' \\ w_{2,0}(\xi,\xi') = \frac{1}{1-\alpha}\left[1 - \xi^{1-\alpha}\right] & \xi' \le \xi \le 1 \end{cases} \qquad (22)$$

Their development on the basis of natural modes $u_n(\xi)$ is easily obtained :

$$w_{1,0}(\xi,\xi') = \sum_n \frac{P_n^2}{\mu_n \Omega_n^2} V_n(\xi')\, u_n(\xi) \qquad (23)$$

$$w_{2,0}(\xi,\xi') = \sum_n \frac{P_n^2}{\mu_n \Omega_n^2} u_n(\xi')\, u_n(\xi) \qquad (24)$$

For i = 1 or 2 and $l \ge 1$, let us define the following set of functions :

$$w_{i,l}(\xi,\xi') = \int_0^1 w_{i,l-1}(\xi'',\xi')\, w_{2,0}(\xi'',\xi)\, d\xi'' \qquad (25)$$

$$W_{i,l}(\xi') = \int_0^1 w_{i,l}(\xi,\xi')\, w_{i,0}(\xi,\xi')\, d\xi \qquad (26)$$

From Eqs. (23) and (24), and orthogonality of modes, it is readily demonstrated that :

$$w_{1,l}(\xi,\xi') = \sum_n \frac{P_n^2}{\mu_n \Omega_n^{2l+2}} V_n(\xi')\, u_n(\xi) \qquad (27)$$

$$w_{2,l}(\xi,\xi') = \sum_n \frac{P_n^2}{\mu_n \Omega_n^{2l+2}} u_n(\xi')\, u_n(\xi) \qquad (28)$$

From which it follows, using Eq. (26)

$$W_{1,l}(\xi') = \sum_n \frac{P_n^2}{\mu_n \Omega_n^{2l+4}} V_n^2(\xi') \qquad (29)$$

$$W_{2,l}(\xi') = \sum_n \frac{P_n^2}{\mu_n \Omega_n^{2l+4}} u_n^2(\xi') \qquad (30)$$

and from Eq. (20)

$$\begin{cases} \sum_n P_n^2 u_n^2(\xi) / \Omega_n^{2l+2} = (2-\alpha)\, W_{2,l}(\xi) & (31) \\ \sum_n P_n^2 V_n^2(\xi) / \Omega_n^{2l+2} = (2-\alpha)\, W_{1,l}(\xi) & (32) \end{cases}$$

Whence it is finally concluded, from Eqs (13) - (15), that for a constant acceleration spectrum $Sa(\omega) = A_o$ :

$$u_m(\xi) = \frac{A_o}{\omega_o^2}\left[(2-\alpha)\, W_{2,1}(\xi)\right]^{1/2} \qquad (33)$$

$$V_m(\xi) = m_o A_o\left[(2-\alpha)\, W_{1,1}(\xi)\right]^{1/2} \qquad (34)$$

$$a_m(\xi) = A_o\left[(2-\alpha)\, W_{2,-1}(\xi)\right]^{1/2} \qquad (35)$$

The SRSS problem for constant acceleration spectrum has thus been reduced to closed form. From the definitions (21), (22), 25, (26) the following formulas are obtained, after some algebra :

$$F_u(\alpha,\xi) = (2-\alpha)\, W_{2,1}(\xi) = \frac{8-3\alpha}{(2-\alpha)(3-\alpha)(5-3\alpha)} - \frac{4\xi^{2-\alpha}}{(2-\alpha)(3-2\alpha)} + \frac{(4-\alpha)\xi^{4-2\alpha}}{(1-\alpha)(2-\alpha)(3-\alpha)} - \frac{6(2-\alpha)\xi^{5-3\alpha}}{(1-\alpha)(3-\alpha)(3-2\alpha)(5-3\alpha)} \qquad (36)$$

$$F_V(\alpha,\xi) = (2-\alpha)\, W_{1,1}(\xi) = \frac{2-\alpha}{1-\alpha}\left[\xi^2 - \frac{2}{3-\alpha}\xi^{3-\alpha}\right] \qquad (37)$$

$$F_a(\alpha,\xi) = (2-\alpha)\, W_{2,-1}(\xi) = \frac{2-\alpha}{1-\alpha}\left[1 - \xi^{1-\alpha}\right] \qquad (38)$$

These expressions are valid if $\alpha \ne 1$, $\alpha \ne \frac{3}{2}$, $\alpha \ne \frac{5}{3}$

For $\alpha = 1$ (homogeneous triangular dam) one obtains :

$$F_u(1,\xi) = \frac{5}{4} - 4\xi + \frac{11}{4}\xi^2 - \frac{3}{2}\xi^2 \operatorname{Ln}\xi \qquad (39)$$

$$F_V(1,\xi) = \frac{1}{2}\xi^2 - \xi^2 \operatorname{Ln}\xi \qquad (40)$$

$$F_a(1,\xi) = -\operatorname{Ln}\xi \qquad (41)$$

## 5 SRSS SOLUTIONS FOR POWER LAW SHEAR BEAMS IN CASE OF A TYPICAL REGULATORY SPECTRUM

Typical regulatory spectra usually comprise :

- a constant-acceleration plateau $S_a = A_o$ for $0 \le T \le T_o$,

- a sloping branch $S_a = A_o (T_o/T)^\beta$ for $T > T_o$ (with $\beta = 1$ or $2/3$ in most cases).

For such a spectrum the SRSS solutions previously determined (Eqs (33) - (35) are valid if all the modes are located in the plateau, i.e. if the following condition is satisfied :

$$\frac{\Omega_1}{2\pi} \omega_o T_o \geq 1 \qquad (42)$$

If the first mode is located in the sloping branch (with the higher modes remaining in the plateau), formulas (33) - (35) can easily be corrected ; for instance, noting $A_1$, the spectral acceleration of the first mode, one has for displacements :

$$\omega_o^4 \; u_m^2(\xi) = A_o^2 (2-\alpha) W_{2,1}(\xi) - \left(A_o^2 - A_1^2\right) P_1^2 \, u_1^2 \, / \Omega_1^4$$

SRSS solutions (33) - (35) can thus be rewritten as follows :

$$\begin{cases} u_m(\xi) = \dfrac{A_o}{\omega_o^2} \left[F_u(\alpha, \xi) - \dfrac{\tau}{\Omega_1^4} P_1^2 \; u_1^2(\xi)\right]^{1/2} & (43) \\ V_m(\xi) = m_o A_o \left[F_V(\alpha, \xi) - \dfrac{\tau}{\Omega_1^4} P_1^2 \; V_1^2(\xi)\right]^{1/2} & (44) \\ a_m(\xi) = A_o \left[F_a(\alpha, \xi) - \tau P_1^2 \; u_1^2(\xi)\right]^{1/2} & (45) \end{cases}$$

with :

- $F_u$, $F_V$, $F_a$ given by Eqs (36) - (41)
- $\Omega_1 = \dfrac{2-\alpha}{2} \zeta_1$, ($\zeta_1$ first zero of Bessel function $J_\lambda$, $\lambda = \dfrac{\alpha - 1}{2-\alpha}$)
- $\tau = 0$ if Eq. (42) is satisfied ; if not $\tau = 1 - \left(\dfrac{\Omega_1}{2\pi} \omega_o \; T_o\right)^{2\beta}$

$$- \; p_1 u_1(\xi) = \frac{2}{\zeta_1 \, J_{\lambda+1}(\zeta_1)} \xi^{\frac{1-\alpha}{2}} J_\lambda\left(\zeta_1 \, \xi^{\frac{2-\alpha}{2}}\right) \qquad (46)$$

$$- \; p_1 V_1(\xi) = \frac{2-\alpha}{J_{\lambda+1}(\zeta_1)} \xi^{1/2} J_{\lambda+1}\left(\zeta_1 \, \xi^{\frac{2-\alpha}{2}}\right) \qquad (47)$$

Good approximate substitutes for Eqs (46) - (47) are given by :

$$p_1 u_1(\xi) \cong \frac{\left(5 - 2\alpha - \xi^{2-\alpha}\right)^3 - 8(2-\alpha)^3}{12(2-\alpha)^2} \qquad (48)$$

$$p_1 V_1(\xi) \cong \frac{\xi}{4(2-\alpha)} \left(5 - 2\alpha - \xi^{2-\alpha}\right)^2 \qquad (49)$$

The corresponding approximation for $\Omega_1$ is :

$$\Omega_1 \cong \left[\frac{(9-4\alpha)(17-6\alpha)}{2(31-12\alpha)}\right]^{1/2} \qquad (50)$$

These approximations are sufficiently accurate for practical purposes, as can be seen from table 1 and 2 where the relative error percentage is computed for the two following cases :

- $\alpha = 0$ (homogeneous column), $\beta = 1$, $\tau = \frac{8}{9}$ ($T_o$ = period of the second mode),
- $\alpha = \frac{4}{3}$ (Gazetas triangular dam), $\beta = 1$, $\tau = \frac{3}{4}$ ($T_o$ = period of the second mode)

**Table 1** : Relative error (%) for $\alpha = 0$, $\beta = 1$, $\tau = \frac{8}{9}$

| $\xi$ | Displacement | Shear Force | Acceleration |
|---|---|---|---|
| 0 | + 1.6 | 0 | + 3.6 |
| 0.2 | + 1.5 | + 2.0 | + 0.8 |
| 0.4 | + 1.1 | + 2.3 | + 0.5 |
| 0.6 | + 0.6 | + 1.7 | + 0.1 |
| 0.8 | + 0.3 | + 0.6 | + 0.0 |
| 1 | 0 | + 0.1 | 0 |

**Table 2** : Relative error (%) for $\alpha = \frac{4}{3}$, $\beta = 1$, $\tau = \frac{3}{4}$

| $\xi$ | Displacement | Shear Force | Acceleration |
|---|---|---|---|
| 0 | + 7.3 | 0 | |
| 0.2 | + 4.7 | + 7.8 | + 1.8 |
| 0.4 | + 2.5 | + 5.3 | + 0.5 |
| 0.6 | + 1.3 | + 2.5 | + 0.1 |
| 0.8 | + 0.8 | + 1.1 | + 0.0 |
| 1 | 0 | + 0.7 | 0 |

Comparison with the response obtained with the first mode alone (as in most simplified methods) shows that the proposed formulas (Eqs (43) - (45) and (48) - (49)) provide a significant improvement for base shear (which is underestimated by 13.3 % for $\alpha = \frac{4}{3}$ in the first mode solution) and a considerable improvement for accelerations (which are very poorly represented in the first mode solution, with relative errors in the range 25 - 70 %). It is also to be noted that the proposed formulas are on the safe side (positive relative error).

The acceleration figure for the first line of table 2 is not given as for $\alpha \geq 1$ the SRSS combination results in a diverging series for accelerations at the top of the beam (see Eq. (38) for $F_a$).

## 6 CONCLUSIONS

Simple closed-form solutions have been obtained for SRSS response of power-law shear beams when the seismic excitation is defined by a typical regulatory spectrum. This result should be of interest for simplified methods (preliminary design of earthfill dams, site effects in alluvium layers, design of piles,...).

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