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An absorbing boundary element for dynamic analysis of two-phase media

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ABSTRACT: Undesireable spurious reflections occur on the fictuous boundaries of the study domain modelled by the Finite Element approach. They affect the computed results and have to be removed from the calculations. Amongst different existing techniques developed for this purpose, the paraxial approximation represents an elegant framework. This approach has been easily extended to the case of two-phase media in this paper. The efficiency of this approximation is verified through special Finite Elements in the case of u-p Biot two-phase dynamic formulation.

1 INTRODUCTION

The Finite Element method is the best adapted numerical approach as far as the non-linear behaviour of the constituting materials has to be considered. Unfortunately, in problems involving wave propagation in infinite media, this technique is vulnerable as the boundaries of the computing mesh cause spurious reflections leading to erroneous results. To remedy this problem, several techniques have been proposed in the literature. In the frequency domain the so-called consistant boundaries work very nicely but are frequency dependant and hard to use in the time domain. In the transitory case Lysmer *et al.* (1969) or White *et al.* (1977) proposed the use of absorbing dampers at the boundary. Smith (1974) proposed an ingenious way to deal with that problem. The paraxial approximation seems to be the most interesting one because of its ability to deal with more inclined incident waves on the boundary by simply refining the approximation (Claerbout *et al.* (1976), Engquist *et al.* (1977) , Clayton *et al.* (1977), Cohen *et al.* (1983)). The first author has already proposed a straightforward way for the development and implementation of paraxial approximation based elements in Finite Element framework (1987).
Paraxial approximation constitutes a local boundary condition which permits diffracting waves to be evacuated from the domain of study. In order to obtain a local transient impedance, a limited development of the wave number is performed on the elastodynamic equations expressed in Fourier domain. For high frequency waves and for the waves impinging the boundary with low inclination this approximation is very accurate. The performance of such boundary elements is proportional to the approximation order. At zeroth order, these boundaries correspond to consistant dampers.
This concept is now generalized into the case of the two-phase media which characterizes the behaviour of saturated soils and rocks. The equations governing the behaviour of two-phase media are derived from the formulation proposed by Biot. A simplifed formulation commonly called u-p formulation, very suitable for practical applications in seismic analysis in which the maximum effective frequency is usually less than 50 Hz has been proposed by Zienkiewicz *et al.* (1984). In this paper we present the absorbing boundary element in the case of this latter formulation. In addition to the continuity requirements for the stress vectors between inner and outer domains the continuity of the hydraulic fluxes have also to be assured. The governing equations presented through the variational formulation give a suitable basis for constructing the weak formulation and the Finite Element implementation of the proposed absorbing boundary.
Finally, an illustrative example of the efficiency of the proposed boundary is presented.

2 GOVERNING EQUATIONS

Several choices of the dependant variables have been proposed in the literature. Here we will limit our presentation to the case of simplified formulations so-called u-p formulation (Zienkiewicz *et al.* 1984) in which the absolute displacements of the solid skeleton and the pore water pressure, will be kept as the primary unknowns.
Assuming Terzaghi's effective stress postulate the governing equations in this case are given as:

$$\mathbf{Div}\ \sigma' - \mathbf{grad}\ p + \rho\ \mathbf{g} = \rho\ \partial_{tt}\mathbf{u_s}$$
$$\operatorname{div}\partial_t\mathbf{u_s} - \operatorname{div}(\mathbf{K}\ \mathbf{grad}\ (p - \rho_f\ \mathbf{g}\cdot\mathbf{x})) - \operatorname{div}(\mathbf{K}\rho_f\partial_{tt}\mathbf{u_s}) + n\partial_t p/K_f = 0 \tag{1}$$

with: $\rho = (1-n)\,\rho_s + n\ \rho_f$

ρ_s	density of the solid
ρ_f	density of the fluid phase
$\mathbf{g}$	acceleration of gravity vector
σ'	effective stress tensor
n	porosity
p	pore fluid pressure
$\mathbf{u_s}$	skeleton displacement vector
K_f	compressibility modulus of fluid
$\mathbf{K}$	permeability tensor given by $\mathbf{K} = \mathbf{k}\ /\rho_f\ \mathbf{g}$

We will search for the solution of the above system on an arbitrary plane Σ on which we will define base vectors $\mathbf{e'}_1$, $\mathbf{e'}_2$ and $\mathbf{e}_3$ (see figure 1).
The displacement vector $\mathbf{u_s}$ will be projected on the tangential plane Σ and its normal unit vector $\mathbf{e}_3$ as:

$$\mathbf{u_s} = \mathbf{u'} + u_3\ \mathbf{e}_3 \tag{2}$$

The mechanical behaviour of the skeleton is assumed to be linear elastic:

$$\sigma' = \lambda\ \mathrm{tr}\ [\varepsilon(\mathbf{u_s})]\ \mathbf{I} + 2\mu\ \varepsilon(\mathbf{u_s}) \tag{3}$$

where λ and μ are Lamé coefficients, **I** the unity tensor and ε the strain tensor.
In order to construct the absorbing boundaries from the paraxial approximation we need firstly to define the displacement and pressure expressions. In the following paragraph the solution procedure for the evaluation of these quantities is presented.

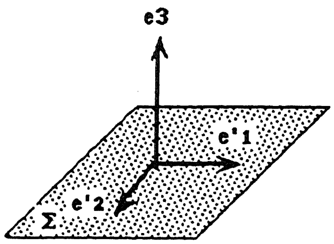

Fig.1: referential on the arbitrary plane Σ

3 SOLUTION PROCEDURE

In this paragraph the analytical solution of the differential equation system given by (1) will be established.
The global equilibrium equation with $\mathbf{u_s}$ and p as major unknowns can be obtained as:

$$(\lambda + \mu)\ \mathbf{grad}\ (\mathrm{div}\ \mathbf{u_s}) + \mu\, \Delta\, \mathbf{u_s} - \mathbf{grad}\ p = \rho\, \partial_{tt}\, \mathbf{u_s} \qquad (4)$$

The gradient and laplace operators for the decomposed displacement vectors are given as:

$$\mathbf{grad}\ (\mathrm{div}\ \mathbf{u_s}) = [\partial_{\mathbf{x'x'}}\mathbf{u'} + \partial_{\mathbf{x'}3}\, u_3] + [\partial_{\mathbf{x'}3}\mathbf{u'} + \partial_{33} u_3]\ \mathbf{e}_3$$

$$\Delta \mathbf{u_s} = [\partial_{\mathbf{x'x'}}\mathbf{u'} + \partial_{33}\mathbf{u'}] + [\ \partial_{\mathbf{x'x'}}\, u_3 + \partial_{33} u_3\ ]\ \mathbf{e}_3$$

The equilibrium equation (2) is then equivalent to the next set of equations:

on the plane Σ:

$$(\lambda + \mu)\ [\partial_{\mathbf{x'x'}}\mathbf{u'} + \partial_{\mathbf{x'}3}\, u_3] + \mu\ [\partial_{\mathbf{x'x'}}\mathbf{u'} + \partial_{33}\mathbf{u'}] - \partial_{\mathbf{x'}}\, p - \rho\, \partial_{tt}\, \mathbf{u'} = 0 \qquad (5)$$

following $\mathbf{e}_3$ unit vector:

$$(\lambda + \mu)\ [\ \partial_{\mathbf{x'}3}\, \mathbf{u'} + \partial_{33}\, u_3\ ] + \mu\ [\ \partial\, u_3 + \partial_{33}\, u_3\ ] - \partial_3\, p - \rho\, \partial_{tt}\, u_3 = 0 \qquad (6)$$

Using the projection relations, the hydraulic equation becomes:

$$\partial_t[\partial_{\mathbf{x'}}\mathbf{u'} + \partial_3 u_3] - [K'\, \partial_{\mathbf{x'x'}} p + K_3\, \partial_{33} p] + n/K_f\, \partial_t p - \rho_f\, [K'\, \partial_{\mathbf{x'}}\partial_{tt}\mathbf{u'} + K_3\, \partial_3\partial_{tt} u_3] = 0 \qquad (7)$$

where K' and K_3 are the components of the permeability tensor with respect to the plane Σ.
The solution of the system of equations given by (5), (6) and (7) is not straightforward as the second and third spacial and temporal derivatives of unknown displacement and pressure fields have to be calculated. Therefore it is easier to perform a Fourier Transform of the above equations with respect to $\mathbf{x'}$ and t. In that way the differential equations transform to partial differential equations on x_3 and the analytical solution can be obtained assuming the plane wave pattern.

on the plane Σ:

$$(C_p^2 - C_s^2)[-\xi'\hat{\mathbf{u}}' + i\, \partial_3 \hat{u}_3]\ \xi' - i/\rho\, \hat{p}\, \xi' + C_s^2[-|\xi'|^2 + \partial_{33}]\hat{\mathbf{u}}' + \omega^2 \hat{\mathbf{u}}' = 0 \qquad (8)$$

following the normal e_3:

$$(C_p^2 - C_s^2)[i\, \xi'\partial_3\hat{\mathbf{u}}' + \partial_{33}\hat{u}_3] + C_s^2[-|\xi'|^2 + \partial_{33}]\hat{u}_3 + \omega^2\, \hat{u}_3 - 1/\rho\, \partial_3\, \hat{p} = 0 \qquad (9)$$

C_s and C_p are respectively the shear and dilatational wave velocities in the medium in the absence of water given by:

$$C_s^2 = \mu / \rho \qquad \text{and} \qquad C_p^2 = (\lambda + 2\mu) / \rho$$

$\hat{\mathbf{u}}'$, $\hat{u}_3$ and $\hat{p}$ designate the Fourier Transforms of displacement components ($\mathbf{u}'$ and u_3) and the pore pressure. ξ' and ω are respectively the wave vector and the angular frequency.
After the transformation the hydraulic equation (7) becomes:

$$\omega\, (-1 + i\rho_f\omega K')\ \xi'\hat{\mathbf{u}}' + \omega\, (i + \rho_f\omega K_3)\partial_3\hat{u}_3 + [K'|\xi'|^2 + i\omega/Q - K_3\partial_{33}]\hat{p} = 0 \qquad (10)$$

Henceforward in this text the permeability will be supposed to be isotropic and frequency independent:

$$K = K' = K_3$$

In order to resolve the equation system, the $\hat{\mathbf{u}}'$ will be decomposed with respect to the plane waves as:

$$\hat{\mathbf{u}}' = [\ \xi'\hat{\mathbf{u}}' / |\xi'|^2\ ]\ \xi' + [\ (\xi', \hat{\mathbf{u}}', e_3) / |\xi'|^2\ ]\ \xi' \wedge e_3 \qquad (11)$$

in which the vectors ξ' and $\xi' \wedge e_3$ are orthogonal.
Three waves are distinguished after the resolution of the above system. The velocities of these waves are given after some simplified hypotheses as:

$$\begin{cases} V_{p1}^2 = C_p^2[1 + K_f(\lambda + 2\mu)/n] \\ V_{p2}^2 = i\omega K K_f[(\lambda + 2\mu)/(\lambda + 2\mu + K_f/n)]/n \\ V_s^2\ = C_s^2 \end{cases} \qquad (12)$$

Therefore, when the pore water's compressibility is very small compared to the compliance $(1/(\lambda+2\mu))$ of the solid skeleton two dilatational waves propagate in the porous medium:
A dilatational wave called P1 propagating with the velocity V_{p1} independent from the frequency. Its definition is close to C_p with a complimentary term proportional to the compressibility of the pore-water,
The second dilatational wave is an attenuating wave with a velocity proportional to the square root of the permeability and the angular frequency.
On the contrary, in the case of high compressible pore-fluid (compared to the compliance of the solid skeleton), the first dilatational wave propagates with a velocity equivalent to that of the classical dialatational waves in one-phase media and the second one is an attenuating wave propagating with a velocity proportional to the permeability, the compressibility of the pore-fluid and the frequency.
The solutions of the differential equations can now be computed as:

$$\xi'\hat{\mathbf{u}}' = |\xi'|\,[\ A_s \exp(-i\, \xi_s\, x_3) + A_{p1} \exp(-i\, \xi_{p1}\, x_3) + A_{p2} \exp(-i\, \xi_{p2}\, x_3)] \qquad (13)$$

$$\hat{u}_3 = A_s |\xi'| / \xi_s \exp(-i\, \xi_s x_3) - A_{p1} \xi_{p1} / |\xi'| \exp(-i\, \xi_{p1} x_3)$$

$$- A_{p2}\, \xi_{p2} / |\xi'| \exp(-i\, \xi_{p2}\, x_3) \qquad (14)$$

$$\hat{p} = -i\, \rho\, \omega^2 / |\xi'| \, [\, (1 - C_p^2 / V_{p1}^2)\, A_{p1} \exp(-i\, \xi_{p1}\, x_3) +$$

$$(1 - C_p^2 / V_{p2}^2)\, A_{p2} \exp(-i\, \xi_{p2}\, x_3)\,] \qquad (15)$$

with:

$$\begin{cases} \xi_{p1}^2 = \omega^2 / V_{p1}^2 - |\xi'|^2 \\ \xi_{p2}^2 = \omega^2 / V_{p2}^2 - |\xi'|^2 \end{cases} \qquad (16)$$

The above relationship shows that the pore-pressure is not influenced by shear waves. The coefficients A, A_{p1}, A_{p2} et A_s can be determined in function of the boundary conditions.

4 DYNAMIC BOUNDARY CONDITION

In order to clarify our approach, let us consider a semi-infinite domain presented in figure 2. This domain is supposed to be composed of two sub-domains Ω_E (outer domain, unbounded except on the ground surface) and Ω_I (inner domain, bounded). On the interface Σ between these two domains, the continuity equations between the displacement (**u**), the stress (**t**) vectors and the hydraulic flux are given as :

$$\begin{cases} \mathbf{u}_I = \mathbf{u}_E \\ \mathbf{t}_I + \mathbf{t}_E(\mathbf{u}_E) = 0 \\ \Phi_I = \Phi_E \end{cases} \qquad (17)$$

At the vicinity of the boundary Σ, a linear elastic behaviour for the constituting materials is assumed so that the notation $\mathbf{t}_E(\mathbf{u}_E)$ has a meaning. Now, we have to introduce a dynamic impedance on the boundary Σ which as we have recalled earlier, has to be local with respect to time and space, and for this purpose, we use the paraxial approximation.

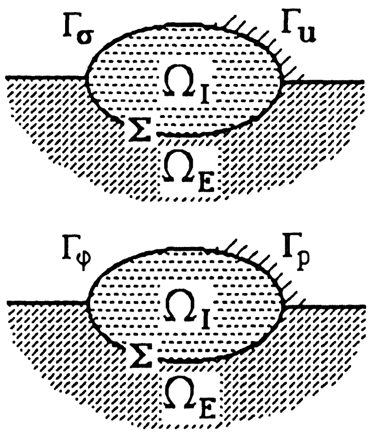

Fig.2 . Schematic representation of the problem

The spectral impedance on the interface represents the spectral action exerted by the outer domain Ω_E on the inner domain Ω_I, when Ω_E is submitted to radiant waves coming from Σ and going to the infinity. To obtain the spectral impedance on the interfaceΣ, we proceed first to the Fourier transformation of the poro-elasto-dynamic equations with respect to time and space vector components in a plane tangent to the boundary. This impedance is not local.
In order to obtain a local transient impedance, a limited development of the wave number is performed. For high frequency waves and for the waves impinging the boundary with low inclination this approximation is very accurate.
To evaluate the impedance on Σ, the stress vector and the hydraulic flux through it have to be calculated on this boundary. To facilitate the presentations this boundary is supposed to be situated at $x_3=0$.
The total stess vector is given by:

$$\mathbf{t}(\mathbf{x}', x_3, t) = \{\, [\, \lambda \operatorname{tr}[\varepsilon(\mathbf{u}_s)] - \mathbf{p}\,]\, \mathbf{I} + 2\mu\, \varepsilon(\mathbf{u}_s)\,\}\, \mathbf{e}_3 \qquad (18)$$

and on the interface Σ, its Fourier Transform with respect to **x'** and t , where $x_3=0$ is given as:

$$\hat{\mathbf{t}}(\xi', x_3, \omega) = [\, i\lambda\, \xi' \hat{u}_0' + (\lambda+2\mu)\partial_3 \hat{u}_{30} - \hat{p}_0\,]\, e_3$$

$$+ \mu\, [\, \partial_3 \xi' \hat{u}_0' / |\xi'|^2 + i\, \hat{u}_{30}]\, \xi'$$

$$+ \mu\, [\partial_3\, (\hat{u}_0', \xi', e_3) / |\xi'|^2 + i\, \hat{u}_{30}\,]\, \xi' \wedge e_3 \qquad (19)$$

or schematically:

$$\hat{\mathbf{t}}(\xi', x_3, \omega) = a_0\, e_3 + b_0 \xi' + c_0\, \xi' \wedge e_3 \qquad (20)$$

5 PARAXIAL APPROXIMATION

let : $\kappa = |\xi'| / \omega$
then, we will adopt the following approximations in order to calculate the stress vector's expression on the boundary:

$$\xi_s = (\omega / C_s)\, (1 - C_s^2\, \kappa^2)^{1/2} \approx (\omega / C_s)\, [\, 1 - 1/2\, (C_s \kappa)^2 + \ldots]$$

$$\xi_{p1} = (\omega / V_{p1})(1 - V_{p1}^2\, \kappa^2)^{1/2} \approx (\omega / V_{p1})[1 - 1/2\, (V_{p1} \kappa)^2 + \ldots]$$

$$\xi_{p2} = (\omega / V_{p2})(1 - V_{p2}^2\, \kappa^2)^{1/2} \approx (\omega / V_{p2})\, [1 - 1/2\, (V_{p2} \kappa)^2 + \ldots]$$

We can see easily that for small values for κ, the waves propoagate in directions close to e_3.
As we have seen in the preceeding sections the velocity of propagating P2 waves is very small compared to V_{p1} and C_p. So, it can be therefore neglected with respect to other quantities. Assuming the zeroth order development, we will obtain:

$$\begin{cases} a_0 = -\, i\, (\lambda+2\mu)\, \omega\, /\, V_{p1}\, \hat{u}_{30} - \hat{p}_0 \\ b_0 = -\, i\, \mu\, \omega /\, C_s\, \xi' \hat{u}_0\, / |\xi'| \\ c_0 = -\, i\, \rho\, C_s\, \omega\, (\hat{u}_0', \xi', e_3)\, / |\xi'| \end{cases} \qquad (21)$$

The total stress vector given in the time domain will be then written as:

$$t_0(t) = (\, -\, \rho\, C_p^2 / V_{p1} \partial_t u_{30} - p_0\,)\, e_3 - \rho\, C_s\, \partial_t u_0' \qquad (22)$$

In the same way the hydraulic flux through Σ can be estimated as:

$$\Phi_0(t) = K\, \rho\, \partial_{tt} u_{30} \qquad (23)$$

In conclusion, we recall that by the paraxial approximation in the outer domain Ω_E, we will obtain a local transient impedance which only introduces the tangent derivatives at the boundary Σ. In the next section, we will retain the following symbolical expressions:

$$\begin{cases} t_E = A(\partial_t u_I) \\ \Phi_E = B(\partial_{tt} u_I) \end{cases} \tag{24}$$

6 VARIATIONAL FORMULATION

Now, let $\mathbf{w}$ be a kinematically admissible virtual displacement field and q an admissible virtual pressure field over Ω_I with boundary Γ given as:

$$\begin{cases} \Gamma = \Gamma_\sigma \cup \Gamma_u \cup \Sigma \\ \Gamma = \Gamma_\varphi \cup \Gamma_p \cup \Sigma \end{cases}$$

Then the governing equations, after the application of the virtual work principle for every w and q as, can be written as:

$$(\rho\partial_{tt}\mathbf{u}_{sI}, \mathbf{w})_{\Omega I} + (\sigma'_I, \varepsilon(\mathbf{w}))_{\Omega I} + (p_I, \mathrm{div}(\mathbf{w}))_{\Omega I} - <t_I,\mathbf{w}>_\Sigma$$

$$= (\rho \mathbf{g},\mathbf{w})_{\Omega I} + <t_I,\mathbf{w}>_\sigma \tag{25}$$

$$(K\rho_f\,\partial_{tt}\mathbf{u}_{sI}, \mathbf{grad\,q})_{\Omega I} + (\mathrm{div}\partial_t\mathbf{u}_{sI}, q)_{\Omega I} + (\mathbf{K grad}\ p_I,$$

$$\mathbf{grad q})_{\Omega I} + (n\,\partial_t p_I / K_f, q)_{\Omega I} = -<\Phi_I,q>_\Sigma - <\Phi_I, q>_{\Gamma\Phi} +$$

$$(\mathbf{K\ grad}\rho_f\,\mathbf{g\ x}, \mathbf{grad q})_{\Omega I} \tag{26}$$

in which $(\,,\,)_{\Omega I}$ denotes the integration over the domain Ω_I and $<\,,\,>_\Gamma$ designates the integration on its boundary Γ.
The above equations for the *zeroth order* approximation become :

$$(\rho\,\partial_{tt}\mathbf{u}_{sI}, \mathbf{w})_{\Omega I} + (\sigma'_{sI}, \varepsilon(\mathbf{w}))_{\Omega I} + (p_I, \mathrm{div}(\mathbf{w}))_{\Omega I} +$$

$$<A_0(\partial_t\mathbf{u}_{sI}),\mathbf{w}>_\Sigma = (\rho\ \mathbf{g}, \mathbf{w})_{\Omega I} + <t_I,\mathbf{w}>_\sigma \tag{27}$$

$$(K\rho_f\,\partial_{tt}\mathbf{u}_{sI}, \mathbf{grad\ q})_{\Omega I} + (\mathrm{div}\partial_t\mathbf{u}_{sI}, q)_{\Omega I} +$$

$$(\mathbf{Kgrad}\ p_I, \mathbf{grad q})_{\Omega I} + (n\,\partial_t p_I / K_f, q)_{\Omega I} + <B_0(\partial_{tt}\mathbf{u}_{sI}), \mathbf{w}>_\Sigma$$

$$= -<\Phi_I, q>_{\Gamma\Phi} + (\mathbf{Kgrad}\rho_f\mathbf{g}, \mathbf{grad\ q})_{\Omega I} \tag{28}$$

7 ILLUSTRATIVE EXAMPLE

To illustrate the efficiency of the proposed absorbing boundary we will consider three soil columns submitted to a Heavy-side type loading on the top. Two 100 m high columns differing only with respect to the boundary condition at the base and a 300 m high column, representing the reference semi-infinite soil domain (as the time window is chosen enough small to prevent the propagating wave arriving at the base) are considered. The geometry and other boundary conditions are presented in figure 3. The hydro-mechanical properties are given in table 1. An implicit Newmark scheme is utilized for the time integration of governing equations.
Vertical displacements and water-pore pressures calculated at three points (respectively at 25, 50 and 100 m from the top) are presented in figures 4, 5 and 6. In all cases some numerical oscillations are observed in the pore pressure diagrams. No numerical oscillations appear in the computed displacement fields.
The computed results show clearly the performance of the paraxial elements in this one-dimensional test. However, small reflections (figure 6) occur in the pore pressure diagram when the paraxial elements are used. It has to be recalled that on the contrary to the one-phase paraxial elements which represent exact absorbing boundaries in one-dimensional problems, the two-phase paraxial elements are approximative due to the adopted simplifications presented above.

Table1: Hydro-mechanical properties for all tests.

Property	Designation	Numerical value
Total density	ρ (Kg/m^3)	1620
Compressibility	K_f (Pa^{-1})	10^{+6}
Permeability	K(m/s)	10^{-7}
Young Modulus	E (Pa)	4.10^8
Poisson Coefficient	ν	0.3

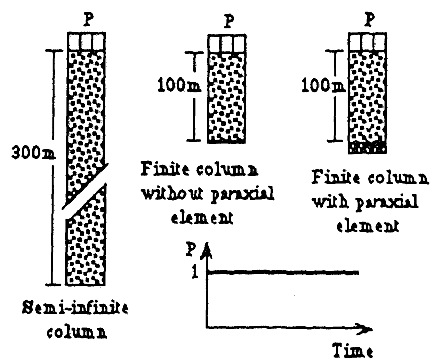

Fig. 3 Geometry and boundary conditions for the test problem

8 CONCLUSIONS

The analysis of the wave propagation in a two-phase medium modelled by the simplified Biot formulation is of a great interest in many Earthquake Engineering problems. Before giving the formulation of the absorbing boundaries we have shown the existance of three plane waves; one shear and two dilatational waves. The velocities calculated for SH and SV waves are identical and equal to the velocity of a shear wave propagating in an equivalent one-phase medium. The displacement components induced by the first dilatational wave act in the same direction, while for the second dilatational as well as the shear waves they are in opposite directions. The second dilatational wave is strongly

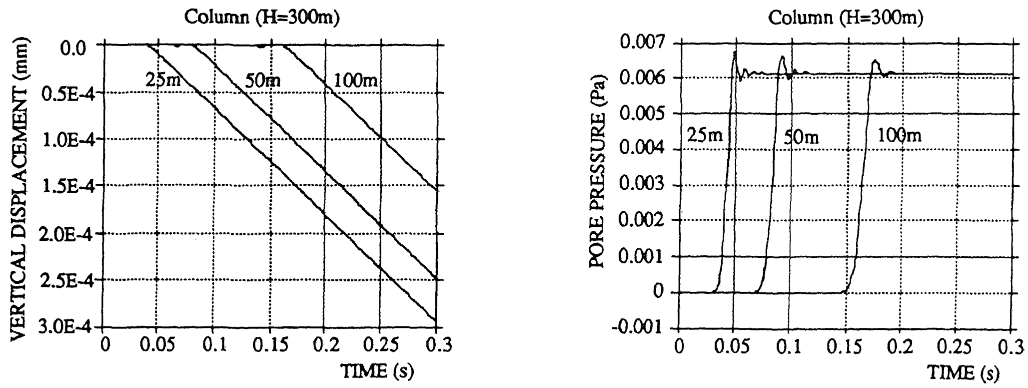

Fig. 4 vertical displacement and pore pressure for semi-infinite column

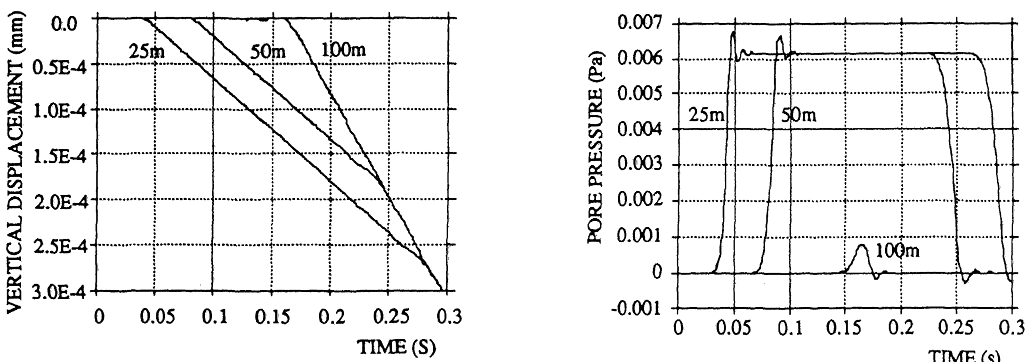

Fig. 5 vertical displacement and pore pressure for finite column without paraxial element

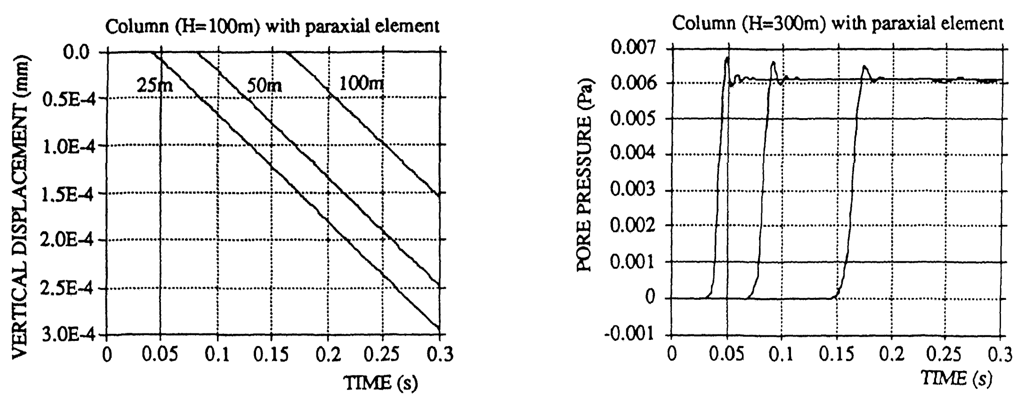

Fig. 6 vertical displacement and pore pressure for finite column with paraxial element

attenuating and negligible compared to the first dilatational wave. The paraxial approximation enables us to construct the absorbing boundaries. The efficiency of these boundaries is verified through an illustrative example. We have also developed following the same methodology paraxial elements in the case of full Biot's formulation which will be presented in a coming publication.

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