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Spectral estimation of bivariate non-stationary processes

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ABSTRACT: A new technique is developed for the estimation of auto-spectra and cross-spectra of multivariate non-stationary random processes which are used as models of earthquake accelerograms. The spectra are estimated by considering the statistical moments of the energy of linear systems (filters) excited by the stochastic seismic processes.
The usefulness of the proposed technique is demonstrated by a number of applications involving both simulated and measured data. For the simulated data, the estimated values of the spectral parameters are found in good agreement with the theoretical ones.

1 Introduction

Stochastic modeling of strong ground accelerograms has already been established as a useful option for predicting and mitigating earthquake related hazards. Essential to this modeling approach, is the availability of reliable models which can be used to simulate artificial earthquakes involved in analytical studies. A large body of research already exists which deals with extracting such information from measured earthquake records. Most of this research, however, treats the seismic accelerograms as univariate stationary processes. Multivariate models of earthquakes have been treated with the simplifying assumption of the various components being uncorrelated. The nonstationary aspect, on the other hand, has received a great deal of attention from various researchers. The main focus has been the estimation of time dependent envelope functions used in modeling evolutionary processes. Typically, a mathematical form of this envelope has to be adopted prior to its identification. A technique is described herein which allows for the estimation of nonstationary spectral density functions without having to assume a particular form for the envelope function. It is an improvement of the technique in Spanos et.al (1987). Further, approaches for estimating quantities related to multivariate processes, such as cross-spectra and quad-spectra are presented.

2 Estimation of Evolutionary Auto-Spectra

Consider the single-degree-of-freedom oscillator described by the following equation of motion

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2 x = f(t), \tag{1}$$

where $\zeta \ll 1$ and ω_0 are its damping ratio and natural frequency, respectively. The symbol $f(t)$ stands for a nonstationary excitation process. Further, let x_1 denote x, and x_2 denote $\dot{x}$. Then, it can be shown that oscillator response can be expressed as

$$x_j(t) = \int_0^t h_j(t-\tau)f(\tau)d\tau \tag{2}$$

where h_j are the impulse response functions of the system. They are given by the equations,

$$h_1(t) = (1/\omega_1)e^{-\zeta\omega_0 t}\sin\omega_1 t \tag{3}$$

and

$$h_2(t) = \frac{d}{dt}h_1(t) \approx e^{-\zeta\omega_0 t}\cos\omega_1 t\ , \tag{4}$$

where $\omega_1 = \sqrt{1-\zeta^2}\omega_0$ denotes the damped natural frequency of the system.

The nonstationary feature of the excitation process $f(t)$ is accounted for by introducing a deterministic modulating function $A(t,\omega)$ (Priestley, 1973) in the form

$$f(t) = \int_{-\infty}^{\infty} A(t,\omega)e^{-i\omega t}dZ(\omega)\ , \tag{5}$$

where $Z(\omega)$ is a process with independent increments. Since $f(t)$ is a real function, its spectral density function is an even function of frequency, and therefore the following relation holds for the envelope function

$$A(t,\omega) = A^*(t,-\omega). \quad (6)$$

Furthermore, it can be shown that the time dependent spectral density function of the excitation process $f(t)$ is given by the equation

$$S(t,\omega) = |A(t,\omega)|^2 S(\omega) , \quad (7)$$

where $S(\omega)$ is a time-independent function of frequency. Using equation (2) in conjunction with equations (5) and (7) gives,

$$<x_j(t)x_k(t)> = \int_{-\infty}^{\infty} S(\omega)\tilde{H}_j(t,\omega)\tilde{H}_k^*(t,\omega)d\omega \quad (8)$$

where

$$\tilde{H}_j(t,\omega) = \int_0^t h_j(\tau)A(t-\tau,\omega)e^{-i\omega\tau}d\tau \quad (9)$$

Performing integration by parts on the last integral, and assuming that $A(0,\omega)$ is equal to 0, yields

$$\tilde{H}_j(t,\omega) = -A(t,\omega)H_j(0,\omega) + \int_0^t \dot{A}(t-\tau,\omega)H_j(\tau,\omega)d\tau , \quad (10)$$

where

$$H_j(\tau,\omega) = \int h_j(\tau)e^{-i\omega\tau}d\tau . \quad (11)$$

Assuming that most of the contribution to the integral in equation (8) is concentrated around ω_0, and after substantial algebraic manipulations (Tein, 1992), the following expression is obtained for the ensemble mean squared values of the response,

$$<x_j^2(t)> = \frac{\pi S(\omega_0)\omega_0^{2j-2}}{\omega_0^2} \int_0^t |A(t-\tau,\omega_0)|^2 e^{-2\zeta\omega_0\tau}d\tau . \quad (12)$$

Consider the total energy $E(t)$ of the oscillator,

$$E(t) = \frac{1}{2}\omega_0^2 x^2(t) + \frac{1}{2}\dot{x}(t) . \quad (13)$$

Then, because of equation (12) one derives,

$$<E(t)> = \pi S(\omega_0)\int_0^t |A(t-\tau,\omega_0)|^2 e^{-2\zeta\omega_0\tau}d\tau . \quad (14)$$

The above equation can be rewritten as the following differential equation,

$$<\dot{E}(t)> + 2\zeta\omega_0<E(t)> = \pi A(t,\omega_0)S(\omega_0) . \quad (15)$$

This equation has also been derived in Spanos et.al (1987) relying on the Fokker-Planck equation associated with Markovian processes. The evolutionary spectrum can therefore be estimated by the expression,

$$S(t,\omega_0) = A(t,\omega_0)S(\omega_0) = \frac{1}{\pi}\left[<\dot{E}(t)> + 2\zeta\omega_0<E(t)>\right] . \quad (16)$$

The above equation provides an estimate of the evolutionary spectral density function of a process $f(t)$, at a frequency ω_0, based on observations of the response to this process of a single-degree-of-freedom oscillator with natural frequency equal to ω_0. The above procedure can be repeated for various values of ω_0, to sweep the frequency domain with a desired accuracy.

Figure (1) shows a target evolutionary Kanai-Tajimi spectrum and its estimates. Simulated time histories corresponding to the target spectral density function are used as excitation to a lightly damped system. In addition to the procedure described above, results corresponding to estimation using only the energy of the response process, that is $\dot{E}(t) = 0$ in equation (16), are shown. It is seen that the inclusion of the time derivative of the energy has improves the estimated spectral density functions. It is emphasized that the estimation procedure based on equation (16) does not involve modeling the modulating function $A(t,\omega)$. Thus it eliminates assumptions regarding this envelope, which may not be accurate. This is also true for errors that are usually associated with the smoothing of the estimated modulating function $A(t,\omega)$.

3 Multi-Variate Estimation of Evolutionary Cross-Spectra

3.1 Models for Evolutionary Cross-Spectral Density Functions

Consider a bivariate process $\{(X(t),Y(t))\}$, in which each component is a nonstationary random process. These processes can be represented as

$$X(t) = \int_{-\infty}^{\infty} A_x(t,\omega)e^{-i\omega t}dZ_X(\omega) , \quad (17)$$

and

$$Y(t) = \int_{-\infty}^{\infty} A_y(t,\omega)e^{-i\omega t}dZ_Y(\omega) , \quad (18)$$

where $Z_X(\omega)$ and $Z_Y(\omega)$ are two Gaussian processes with independent increments satisfying the relationship

$$<dZ_X(\omega)dZ_Y(\omega)> = S_{XY}(\omega)\delta(\omega)d(\omega) . \quad (19)$$

The cross-covariance function of the two random processes can be expressed by the equation,

$$\begin{aligned} <X(t)Y(t)> &= \int_{-\infty}^{\infty} A_x(t,\omega)A_y^*(t,\omega)S_{XY}(\omega)d\omega \\ &= \int_{-\infty}^{\infty} S_{XY}(t,\omega)d\omega . \end{aligned} \quad (20)$$

Note that the cross-spectral density function $S_{XY}(t,\omega)$ defined here lends itself to a physical interpretation similar to that of the cross-spectral density function of a bivariate stationary process, which represents the average value of the product of the intensity of the two processes with respect to frequency. However, in the nonstationary case these intensities are time dependent, and accordingly, the cross-spectral density function is also time dependent. In general, the stationary cross-spectral density function $S_{XY}(\omega)$ is a complex function and has the following properties

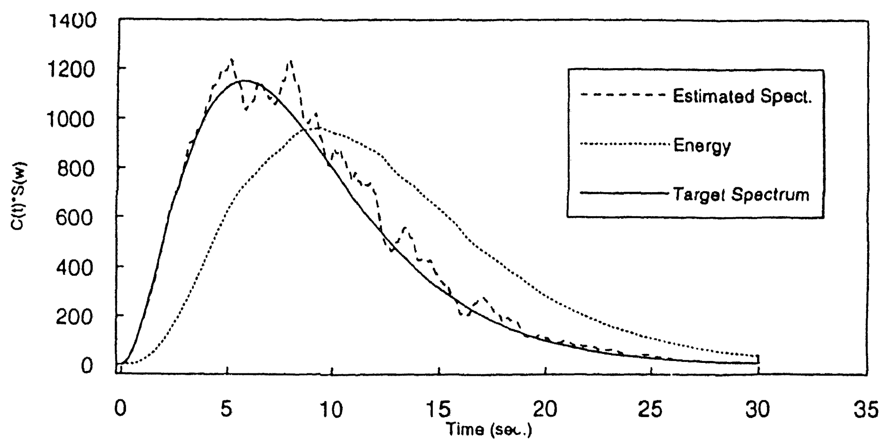

Figure 1 Evolutionary K-T Spectrum

$$S_{XY}(-\omega) = S^*_{XY}(\omega) = S_{YX}(\omega)\ , \tag{21}$$

and can therefore be expressed as

$$S_{XY}(\omega) = C_{XY}(\omega) - iQ_{XY}(\omega)\ . \tag{22}$$

In the above equation, $C_{XY}(\omega)$, called the co-spectrum, is a real-valued even function of ω, and $Q_{XY}(\omega)$,called the quad-spectrum, is a real-valued odd function of ω.

3.2 Approximate Analysis of Evolutionary Cross-spectral Density Functions

A procedure is now developed for estimating nonstationary cross-spectral density functions of bivariate processes. These processes are viewed as the input to the same lightly damped single-degree-of-freedom oscillator with responses denoted by $x(t)$ and $y(t)$ respectively. The expected value of the responses can be written as

$$<x_j(t)y_k(t)> = \int_{-\infty}^{\infty} S_{XY}(\omega)\tilde{H}_{jx}(t,\omega)\tilde{H}^*_{ky}(t,\omega)d\omega \tag{23}$$

where x_1 and x_2 denote x and $\dot{x}$ respectively, while y_1 and y_2 denote y and $\dot{y}$ respectively. Further,

$$\tilde{H}_{jx}(t,\omega) = \int_0^t h_j(\tau)A_x(t-\tau,\omega)e^{-i\omega\tau}d\tau \tag{24}$$

$$\tilde{H}_{ky}(t,\omega) = \int_0^t h_k(\tau)A^*_Y(t-\tau,\omega)e^{i\omega\tau}d\tau. \tag{25}$$

Similarly to the univariate case, equations (24) and (25) can be integrated by parts with initial conditions $A_x(0,\omega) = 0$ and $A^*_y(0,\omega) = 0$ to obtain,

$$\tilde{H}_{jx}(t,\omega) = -A_x(t,\omega)H_j(0,\omega) + \int_0^t \dot{A}_x(t-\tau,\omega)H_j(\tau,\omega)d\tau \tag{26}$$

and

$$\tilde{H}^*_{ky}(t,\omega) = -A^*_y(t,\omega)H_k(0,\omega) + \int_0^t \dot{A}^*_y(t-\tau,\omega)H_k(\tau,\omega)d\tau. \tag{27}$$

Using the above equations along with the assumption that the contribution to the integrals is concentrated around frequencies $\pm\omega_0$, leads to the following expression for the expected value of the cross-product of the response processes,

$$<x_j(t)y_k(t)> = \frac{\pi}{\omega_0}\left[\frac{1+(-1)^{j+k}}{2}\omega_0^{j+k-3}C_{XY}(\omega_0) - i^{j-k+1}\frac{1-(-1)^{j+k}}{2}Q_{XY}(\omega_0)\right] \times \int_0^t A_x(t-\tau,\omega_0)A^*_y(t-\tau,\omega_0)e^{-2\zeta\omega_0\tau}d\tau. \tag{28}$$

Letting $j = k = 1$ and $j = k = 2$ in the above equation gives

$$<x(t)y(t)> = \frac{\pi}{\omega_0^2}C_{XY}(\omega_0)\int_0^t A_x(t-\tau,\omega_0)A^*_y(t-\tau,\omega_0)e^{-2\zeta\omega_0\tau}d\tau \tag{29}$$

and

$$<\dot{x}(t)\dot{y}(t)> = \pi C_{XY}(\omega_0)\int_0^t A_x(t-\tau,\omega_0)A^*_y(t-\tau,\omega_0)e^{-2\zeta\omega_0\tau}d\tau, \tag{30}$$

respectively. As in the uni-variate case, introduce an energy like quantity,

$$E_{XY}(t) = \frac{1}{2}\left(\omega_0^2 x(t)y(t) + \dot{x}(t)\dot{y}(t)\right)\ . \tag{31}$$

Then, a differential equation is obtained for $<E_{XY}(t)>$. Specifically,

$$\begin{aligned} <\dot{E}_{XY}(t)> + 2\zeta\omega_0<E_{XY}(t)> &= \pi A_x(t,\omega_0)A^*_y(t,\omega_0)C_{XY}(\omega_0) \\ &= \pi C_{XY}(t,\omega_0)\ . \end{aligned} \tag{32}$$

The next step involves estimating the quad-spectra. This can be achieved by letting $j = 1$, $k = 2$ and $j = 2$, $k = 1$ respectively in equation (28) to yield

$$\begin{aligned} <\dot{x}(t)y(t)> &= -<x(t)\dot{y}(t)> \\ &= \frac{\pi}{\omega_0}\int_0^t A_x(t-\tau,\omega_0)A^*_y(t-\tau,\omega_0)e^{-2\zeta\omega_0\tau}d\tau\ . \end{aligned} \tag{33}$$

Defining the following quantity,

$$D_{XY}(t) = \frac{\omega_0}{2}(\dot{x}(t)\, y(t) \;-\; x(t)\, \dot{y}(t)) \;, \qquad (34)$$

and substituting equation (33) into the ensemble average of equation (34) yields,

$$<D_{XY}(t)> = \pi\, Q_{XY}(\omega_0) \int_0^t A_x(t-\tau,\omega_0)A_y^*(t-\tau,\omega_0)e^{-2\zeta\omega_0\tau}d\tau \qquad (35)$$

The differential equation corresponding to equation (35) can be expressed as

$$<\dot{D}_{XY}(t)> + 2\zeta\omega_0 <D_{XY}(t)> \;=\; \pi A_x(t-\tau,\omega_0)A_y^*(t-\tau,\omega_0) \;=\; \pi Q_{XY}(t,\omega_0) \qquad (36)$$

An effective procedure for the nonstationary cross spectral estimation of multivariate earthquake ground motion has been developed with the aid of multifilter techniques (Sawada and Kameda, 1988). The corresponding results were obtained as follows,

$$C_{XY}(t,\omega_0) \;=\; \frac{2\zeta\omega_0}{\pi} <E_{XY}(t)> \qquad (37)$$

and

$$Q_{XY}(t,\omega_0) \;=\; \frac{2\zeta\omega_0}{\pi} <D_{XY}(t)> \qquad (38)$$

These results do not account for the rate of change of the energy function.

Two examples are now developed to demonstrate the effectiveness of the proposed techniques. The first example involves two sets of uncorrelated white noise processes. These are obtained using the following linear combination,

$$\begin{Bmatrix} \bar{X}(t) \\ \bar{Y}(t) \end{Bmatrix} = \begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \begin{Bmatrix} W_1(t) \\ W_2(t) \end{Bmatrix} \qquad (39)$$

where $W_1(t)$ and $W_2(t)$ are two uncorrelated white noise processes with constant spectral density function S_1 and S_2, respectively; the symbols L_1, L_2, L_3 and L_4 denote constants. The modulated correlated processes are

$$\begin{Bmatrix} X(t) \\ Y(t) \end{Bmatrix} = \begin{Bmatrix} C_1(t)\bar{X}(t) \\ C_2(t)\bar{Y}(t) \end{Bmatrix} \qquad (40)$$

Therefore, the analytical expressions for the co-spectrum $C_{XY}(t,\omega_0)$ and quad-spectrum $Q_{XY}(t,\omega_0)$ are

$$C_{XY}(t,\omega_0) \;=\; C_1(t)C_2(t)\;(\;L_1L_3S_1 \;+\; L_2L_4S_2\;) \qquad (41)$$

and

$$Q_{XY}(t,\omega_0) \;=\; 0\;. \qquad (42)$$

In Figure (2), it is seen that both the co-spectrum and the quad-spectrum are very well approximated. The target spectrum itself is well estimated using the proposed technique.

The second example demonstrates the usefulness of the proposed technique for processes having the Kanai-Tajimi spectrum. The equation of motion for this model is given by the equation,

$$\ddot{\bar{Y}} + 2\zeta_g\omega_g\dot{\bar{Y}} + \omega_g^2\bar{Y} \;=\; 2\zeta_g\omega_g\dot{\bar{X}} + \omega_g^2\bar{X} \qquad (43)$$

where $\bar{X}$ is a white noise with constant spectral density function S, and $\bar{Y}$ is the process with Kanai-Tajimi spectrum. Therefore, The frequency response function corresponding to equation (43) is given by the equation,

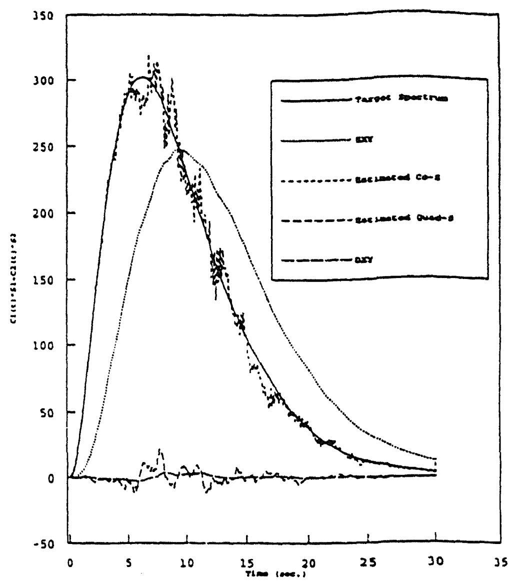

Figure 2 Cross-Spectrum of Evolutionary White Noise

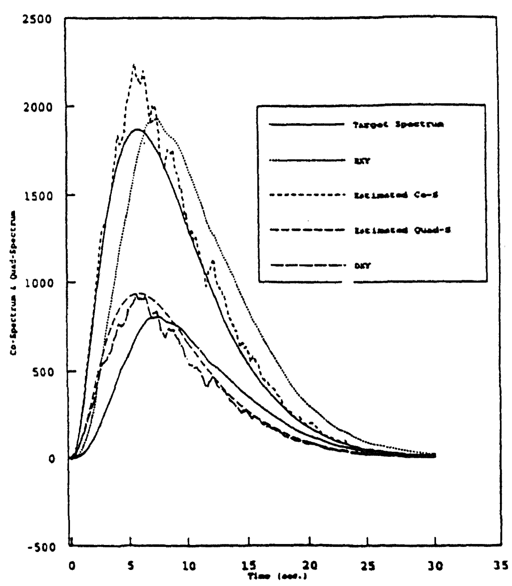

Figure 3 Cross-Spectrum of Evolutionary Broad Band Process

$$H(\omega) \;=\; \frac{\omega_g^2 + 2i\zeta_g\omega_g\omega}{\omega_g^2 - \omega^2 + 2i\zeta_g\omega_g\omega}\;. \qquad (44)$$

The correlated processes are given by equation (40). Therefore, the co- and quad- spectra of X and Y can be approximated using the following expressions

$$C_{XY}(t,\omega_0) \;=\; \frac{\omega_g^2\left(\omega_g^2-\omega^2\right)+4\zeta_g^2\omega_g^2\omega^2}{\left(\omega_g^2-\omega^2\right)^2+4\zeta_g^2\omega_g^2\omega^2}C_1(t)C_2(t)S\;, \qquad (45)$$

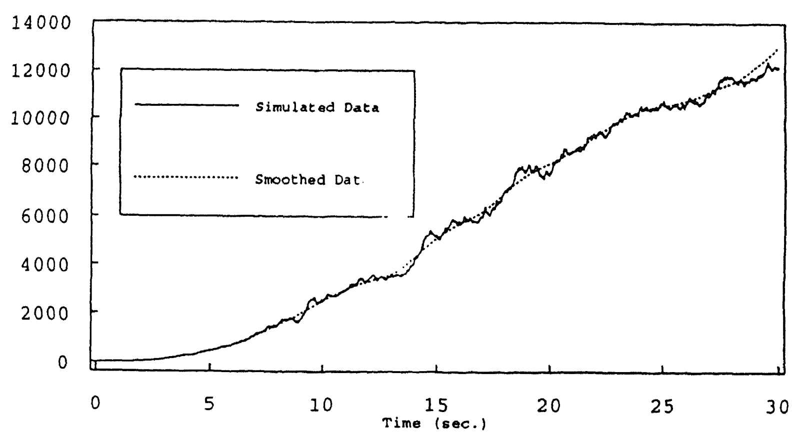

Figure 4 Fitting the Increasing Function - Simulate 50 sets of Data

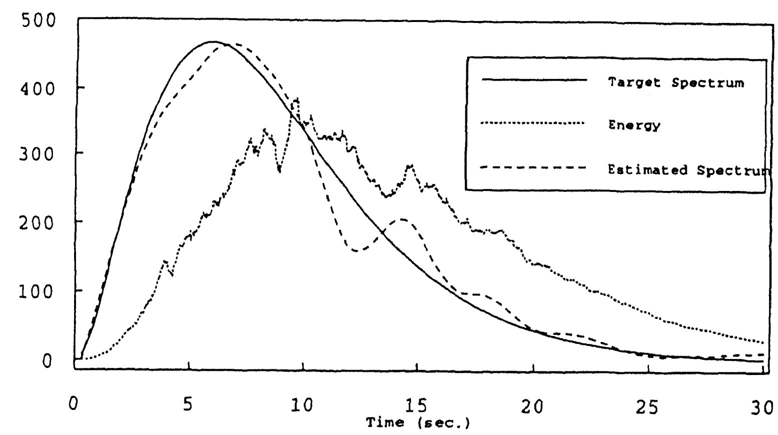

Figure 5 Piecewise Linear Smoothing with Window Size 4 sec.

and

$$Q_{XY}(t,\omega_0) = \frac{2\zeta_g\omega_g\omega^3}{(\omega_g^2-\omega^2)^2+4\zeta_g^2\omega_g^2\omega^2}C_1(t)C_2(t)S\ . \quad (46)$$

Examining Figure (3), it is noted that the estimate of the quad-spectrum matches very well with the analytic solution in equation (46). For the co-spectrum, however, a certain discrepancy is observed. This can be attributed to the numerator in equation (45) being a function of ω^2. This leads to an error in approximating the integrals as given by equations (26)-(30).

4 Smoothing of the Energy Functions

Based on the procedures developed in the previous sections, good estimates can be obtained for the spectral density functions of nonstationary multivariate processes. These estimates, however, are not safeguarded against violating the inherent positive definite property of auto-spectral densities. This deficiency can be attributed to the limited number of records from which spectral density information is extracted. To remedy this problem, the time rate of change of the energy function is forced to have a merely corrective effect on the overall spectrum. This is achieved by fitting a nondecreasing function to the data corresponding to the term $<E(t)>e^{2\zeta\omega_0 t}$. The smoothing is achieved by assuming the spectral density to have a linear variation over successive small time windows. This results in the following approximation

$$<E(t)>e^{2\zeta\omega_0 t} = (b_0(t_0,\omega_0)+b_1(t_0,\omega_0)t)\,e^{2\zeta\omega_0 t}+b_3(t_0,\omega_0)$$
$$t_0-\frac{t_s}{2}\le t\le t_0+\frac{t_s}{2}\ . \quad (47)$$

In the above equation, t_s denotes the time interval. Standard least squares procedures can be used for estimating the coefficients of the linear model. The procedure just described can be the basis for a reliable estimation of the spectral density function of nonstationary processes based on a relatively small number of observed records. Further details of the analysis can be found in Tein (1992). Figures (4) and (5) show the results associated with a simulated non-

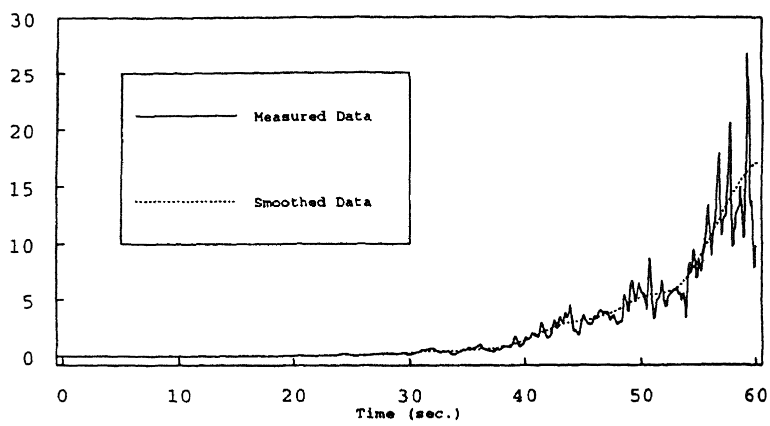

Figure 6 Smoothing of the Energy Term - Average 3 sets of Data

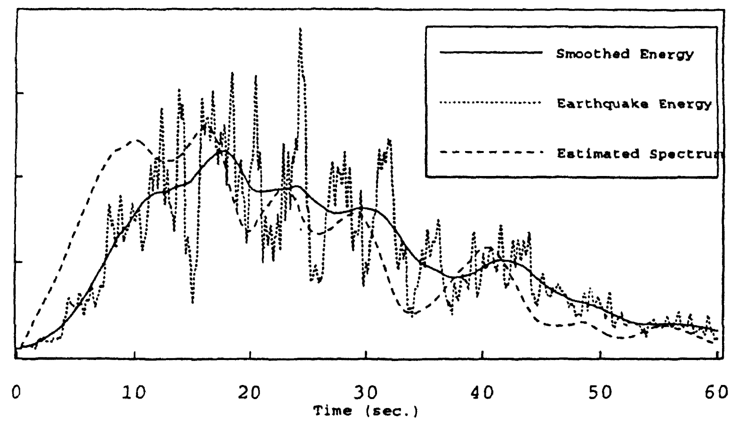

Figure 7 Estimated Spectral Density Function

stationary process with a modulating envelope given by $100t^2e^{-\alpha t}$, where α is a positive number equal to 0.34. Fifty records are used in obtaining the results corresponding to this example. Figure (4) shows the smoothing of the function $<E(t)>e^{2\zeta\omega_0 t}$, while Figure (5) shows the estimated and the target spectral density functions. On the same plot is shown the spectral density function obtained from a multifilter analysis ignoring the rate of change of the energy. Figures (6) and (7) show the results associated with using earthquake records from the 1985 Mexico City earthquake. Again, Figure (6) shows the result of fitting a nondecreasing function to the energy term, while Figure (7) shows the results corresponding to the estimated spectral density function.

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