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Attenuation of waves in ground with fading memory

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ABSTRACT: The fading memory theory is introduced to investigate the mechanism of wave attenuation due to viscosity by considering the ground to be a viscoelastic continuum. In addition, various other dynamic characteristics of ground are discussed based on this concept. The constitutive equation, the wave equation and the corresponding relationship for the frequency dependent wave attenuation index $Q(\omega)$ are presented. The memory functions of the general spring–dashpot model, the creep function model, and the hysteretic model are shown to be special cases of fading memory theory. The method of developing the memory function based on realistic $Q(\omega)$ is also discussed. It is concluded that unlike the dashpot model, the fading memory theory can explain wider range of attenuation properties. To achieve realistic representation of the dynamic characteristics of ground, justifiably pertinent memory function can be selected. The memory function may also be suitably derived from observed data.

1 INTRODUCTION

Viscoelastic material is often represented by spring–dashpot model. Such representation results in an exponential type of attenuation function. However, the wave attenuation characteristics Q^{-1}, which is the dimensionless measure of the internal friction or the anelasticity (e.g. Aki and Richards 1980), is found to be nearly constant for a certain frequency range. Such wave attenuation characteristics cannot be adequately represented by various types of spring–dashpot models widely used in earthquake engineering. For example, the viscoelastic solid of Kelvin–Voigt type can not explain the result obtained from experiment as discussed subsequently. Other types of spring–dashpot systems can offer special Q for soil (Qaisar 1989), but again they are found to be inadequate. Thus there is a need to develop suitable viscoelastic constitutive model that can explain wider range of attenuation characteristics. Such characteristics of Q are necessary for theoretical investigations, such as causality of wave propagation (e.g. Aki and Richards 1980).

The fading memory theory has been well developed in the continuum mechanics. But its application in wave propagation and attenuation has not been attempted. This paper is concerned with the study of wave propagation and attenuation in infinitesimal linear viscoelastic material with fading memory. In fading memory theory, viscoelastic part in the constitutive equation is given by functional form of all the past trace of strain rate, which makes it possible to represent various dynamic characteristics.

2 FADING MEMORY THEORY FOR LINEAR VISCOELASTIC CONTINUUM

The constitutive equation for linear, isotropic material with fading memory is described in functional form using tensor expressions as follows (Eringen 1975, Truesdell 1973, Fillipov 1983, Izumi et al. 1989):

$$\sigma_{kl} = \lambda_e\, \varepsilon_{rr}\, \delta_{kl} + 2\mu_e\, \varepsilon_{kl} + \int_{-\infty}^{t} \left[\lambda_v(t-s) \frac{\partial \varepsilon_{rr}(s)}{\partial s} \delta_{kl} + 2\mu_v(t-s) \frac{\partial \varepsilon_{kl}(s)}{\partial s} \right] ds \tag{1}$$

where the material body is assumed to be initially strain–free. Here, t is the time, σ_{kl} is the stress tensor, ε_{kl} is the infinitesimal strain tensor, λ_e and μ_e are the Lamè's elastic constants. $\lambda_v(t)$ and $\mu_v(t)$ are the memory functions which satisfy the following two conditions:

$$\lambda_v(t) = \mu_v(t) = 0\,, \quad \text{for } t<0$$
$$\lim_{t\to\infty} \lambda_v(t) = \lim_{t\to\infty} \mu_v(t) = 0 \tag{2}$$

The former condition indicates the causality and the latter is concerned with the axiom of memory. The relation between infinitesimal strain tensor and displacement vector is:

$$\varepsilon_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k}) \tag{3}$$

where $u_{k,l} = \partial u_k / \partial x_l$. The balance law of body is:

$$\sigma_{lk,l} + \rho(f_k - a_k) = 0 \tag{4}$$

where ρ is the density, f_k is the body force and a_k is the acceleration. Considering the body force f_k=0, the wave equation is obtained by substituting Eqs.1 and 3 into Eq.4.

$$\begin{aligned}&(\lambda_e + \mu_e) u_{l,lk} + \mu_e u_{k,ll} \\ &+ \int_{-\infty}^{t} \left\{ \left[\lambda_v(t-s) + \mu_v(t-s) \right] \frac{\partial u_{l,lk}}{\partial s} \right. \\ &\left. + \mu_v(t-s) \frac{\partial u_{k,ll}}{\partial s} \right\} ds = \rho \frac{\partial^2 u_k}{\partial t^2}\end{aligned} \tag{5}$$

Considering one dimensional case, the constitutive equation and the wave equation are rewritten as follows:

$$\sigma(t) = A\,\varepsilon(t) + \int_{-\infty}^{t} m(t-s) \frac{d\varepsilon(s)}{ds} ds \tag{6}$$

$$A \frac{\partial^2 u}{\partial x^2} + \int_{-\infty}^{t} m(t-s) \frac{\partial^3 u}{\partial s \partial x^2} ds = \rho \frac{\partial^2 u}{\partial t^2} \tag{7}$$

for normal direction (P-wave)
$A = \lambda_e + 2\mu_e, \quad m(t) = \lambda_v(t) + 2\mu_v(t)$

for shear direction (S-wave)
$A = \mu_e, \quad m(t) = \mu_v(t)$

The memory function $m(t)$ satisfies conditions similar to Eq.2.

$$\begin{aligned} m(t) &= 0, \quad \text{for } t<0 \\ \lim_{t\to\infty} m(t) &= 0 \end{aligned} \tag{8}$$

3 MEMORY FUNCTION FOR DASHPOT MODELS

Until now, some viscoelastic models have been developed and have found their applications in earthquake engineering with the concept of "spring" and "dashpot". Memory functions are derived for such models and it is understood that such models are the special cases of fading memory theory. The general constitutive law for spring-dashpot system can be written as follows (Flügge 1973):

$$\sum_{k=0}^{m} p_k \frac{d^k \sigma}{dt^k} = \sum_{k=0}^{n} q_k \frac{d^k \varepsilon}{dt^k} \tag{9}$$

where p_k and q_k are constants of "spring" and "dashpot". As the special cases of this equation, the Kelvin-Voigt type (m=0, n=1), the Maxwell type (m=1, n=0) and the 3-element type (m=n=1) viscoelastic models are derived. After applying Fourier transform to Eq.9, the following equation is obtained.

$$\Sigma(\omega) = \frac{\sum_{k=0}^{n} q_k (i\omega)^k}{\sum_{k=0}^{m} p_k (i\omega)^k} E(\omega) \tag{10}$$

where Σ(ω) and E(ω) are functions of frequency, ω, and they are obtained by Fourier transform of $\sigma(t)$ and $\varepsilon(t)$, respectively. In the following discussions, we assume that the minuscules with (t) have meaning in time domain while majuscules with (ω) are Fourier transformed functions in frequency domain. Fourier transform of Eq.6 yields:

$$\begin{aligned}\Sigma(\omega) &= A\,E(\omega) + \int_{-\infty}^{\infty} \int_{-\infty}^{t} m(t-s) \frac{\partial \varepsilon(s)}{\partial s} ds\, e^{-i\omega t} dt \\ &= A\,E(\omega) + i\,\omega\,E(\omega)\,M(\omega)\end{aligned} \tag{11}$$

Then the memory function in frequency domain is:

$$M(\omega) = \frac{1}{i\omega} \left\{ \frac{\Sigma(\omega)}{E(\omega)} - A \right\} \tag{12}$$

Using Eqs. 10 and 12, the memory function for spring-dashpot model in time domain is obtained by inverse Fourier transform:

$$\begin{aligned} m(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} M(\omega)\, e^{i\omega t} d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{i\omega} \left\{ \frac{\sum_{k=0}^{n} q_k (i\omega)^k}{\sum_{k=0}^{m} p_k (i\omega)^k} - A \right\} e^{i\omega t} d\omega \end{aligned} \tag{13}$$

For the purpose of comparison, the memory functions of some of the common models are presented in the following. The constitutive equation for the 3-element model is:

$$\sigma(t) + p \frac{d\sigma(t)}{dt} = m_R\, \varepsilon(t) + q \frac{d\varepsilon(t)}{dt} \tag{14}$$

where m_R is a constant. Using A=m_R, the memory function is obtained as follows:

$$m(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{q - p m_R}{1 + i\omega p} e^{i\omega t} d\omega \tag{15}$$

If $p \neq 0$ then Eq.16 yields for the memory function of the 3-element type elastic model:

$$m(t) = \left(\frac{q}{p} - m_R \right) e^{-\frac{t}{p}}\, U(t) \tag{16}$$

where $U(t)$ is the step function. If p=0 (for Kelvin-

Voigt type) then

$$m(t) = q\,\delta(t) \tag{17}$$

where $\delta(t)$ is the delta (impulse) function. If $p \neq 0$ and $m_R=0$ (for Maxwell type) then

$$m(t) = \frac{q}{p} e^{-\frac{t}{p}} U(t) \tag{18}$$

4 MATERIAL CHARACTERISTICS BY MEMORY FUNCTION

Complex modulus of a material is given by a ratio of stress and strain as follows (Fung 1984):

$$\begin{aligned} \Psi(\omega) &= \frac{\Sigma(\omega)}{E(\omega)} = A + i\,\omega M(\omega) \\ &= A - \omega M_I(\omega) + i\,\omega M_R(\omega) \\ &= \Psi_R(\omega) + i\,\Psi_I(\omega) \end{aligned} \tag{19}$$

where $M_R(\omega)$ and $M_I(\omega)$ are real and imaginary parts of $M(\omega)$ respectively. Thus the Q^{-1} is obtained as follows:

$$\frac{1}{Q(\omega)} = \frac{\Psi_I(\omega)}{\Psi_R(\omega)} = \frac{\omega M_R(\omega)}{A - \omega M_I(\omega)} \tag{20}$$

For example, Q^{-1} for the 3-element type model is:

$$\frac{1}{Q(\omega)} = \frac{(q - p m_R)\omega}{m_R + pq\omega^2} \tag{21}$$

for the Kelvin-Voigt type:

$$\frac{1}{Q(\omega)} = \frac{q\omega}{m_R} \tag{22}$$

and for the Maxwell type:

$$\frac{1}{Q(\omega)} = \frac{1}{p\omega} \tag{23}$$

Q^{-1} for the Kelvin-Voigt model is clearly unsuitable to explain relatively constant Q.

The following viscoelastic constitutive equation using creep function (e.g. Aki and Richards 1980) is frequently used:

$$m_U\,\varepsilon(t) = \sigma(t) + \int_{-\infty}^{t} \sigma(\tau)\frac{\partial \phi(t-\tau)}{\partial \tau}d\tau \tag{24}$$

where m_U is a constant relative to Lamé's constants and $\phi(t)$ is a creep function which satisfies:

$$\phi(t) = 0\,, \quad \text{for } t < 0 \tag{25}$$

Comparing eq.11 and Fourier transform of eq.24, and using $A=m_U$, following relations between the memory function and the creep function are obtained.

$$m(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-m_U \Phi(\omega)}{1 + i\omega\Phi(\omega)} e^{i\omega t}d\omega \tag{26}$$

$$\phi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{-M(\omega)}{m_U + i\omega M(\omega)} e^{i\omega t}d\omega \tag{27}$$

5 SOLUTION TO WAVE EQUATION

One dimensional wave equation is:

$$C_0^2 \frac{\partial^2 u}{\partial x^2} + \int_{-\infty}^{t} m_0(t-s)\frac{\partial^3 u}{\partial s \partial x^2}ds = \frac{\partial^2 u}{\partial t^2} \tag{28}$$

where $C_0^2 = A/\rho$, $m_0(t-s) = m(t-s)/\rho$. Considering vibration term in displacement as $u = u\exp(i\omega t)$, Eq.28 yields

$$\left\{C_0^2 + i\,\omega M_0(\omega)\right\}\frac{\partial^2 u}{\partial x^2} = (i\,\omega)^2 u \tag{29}$$

then the displacement solution is

$$u(x,t) = \left[C_f e^{-iB(\omega)x} + C_b e^{iB(\omega)x}\right]e^{i\omega t} \tag{30}$$

where

$$B(\omega)^2 = \frac{\omega^2}{C_0^2 + i\,\omega M_0(\omega)} \tag{31}$$

The phase velocity is

$$V^2 = c_0^2 - \omega M_{0_I}(\omega) \tag{32}$$

If $Q^{-1} \ll 1$ then the solution is approximated as:

$$u(x,t) \approx C_f e^{-\frac{\omega x}{2QV}} e^{i\omega\left(t-\frac{x}{V}\right)} + C_b e^{\frac{\omega x}{2QV}} e^{i\omega\left(t+\frac{x}{V}\right)} \tag{33}$$

It is important to note that $B(\omega)$ and $V(\omega)$ are dependent on the characteristics of material, and are to be determined by memory function.

In continuum mechanics (e.g. Eringen 1975, Truesdell 1973), following two types of memory functions are suggested:

$\exp(-t^p)$ type

$$m(t) = M_R e^{-\left(\frac{(t+a)}{b}\right)^p} U(t)\,, \quad p>0 \tag{34}$$

$1/t^p$ type

$$m(t) = \frac{M_R}{(a+t)^p + b} U(t)\,, \quad p>0 \tag{35}$$

where M_R, a, b and p are all constants. It is noted that the memory function of the 3-element type elastic

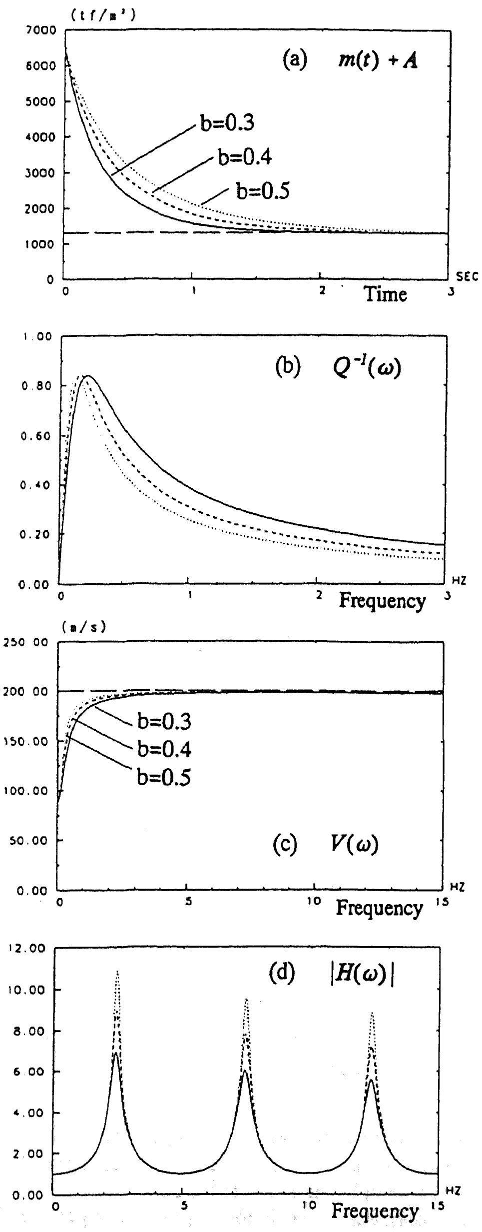

Fig.1 Parameter dependency of memory function

$$m(t) = \alpha e^{-\left(\frac{t+a}{b}\right)^p}, \; a=0, \; p=0.9.$$

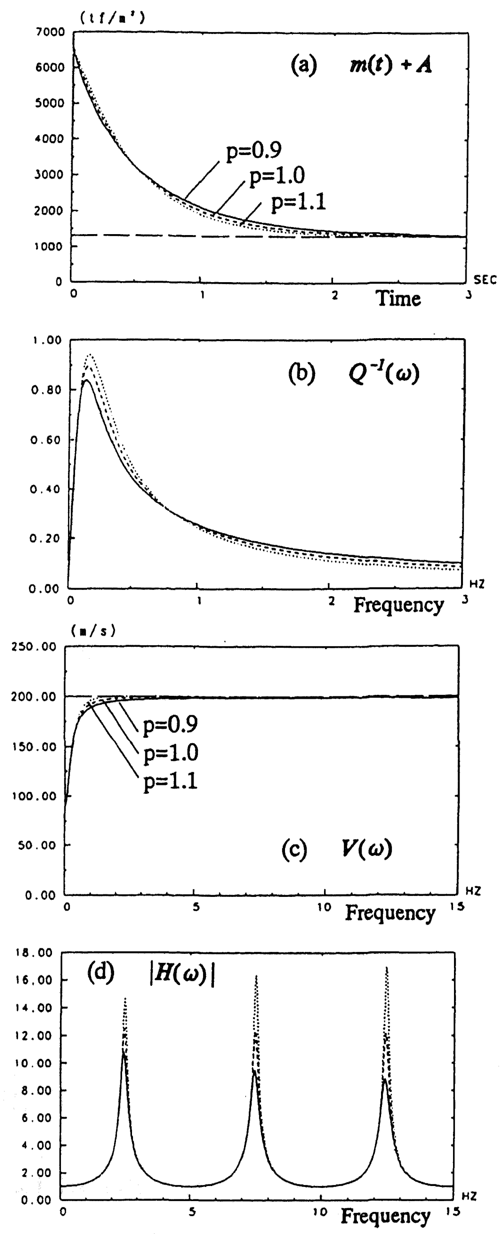

Fig.2 Parameter dependency of memory function

$$m(t) = \alpha e^{-\left(\frac{t+a}{b}\right)^p}, \; a=0, \; b=0.5.$$

model is a special case of Eq.34. Examples are presented for the $\exp(-t^p)$ type memory function with various parameters in Figs. 1 and 2. In each figure, (a) represents the memory function, (b) represents the Q^{-1} according to the memory function in (a), (c) represents the dispersive characteristics of wave velocity and (d) is the transfer function of the 2-layer soil model. It is seen that the viscoelastic model with fading memory can explain various attenuation effects.

6 CALCULATION OF MEMORY FUNCTION FROM Q^{-1}

The relation between Q^{-1} and memory function is obtained by rewriting eq.20 as follows:

$$M_R(\omega) + \frac{1}{Q(\omega)} M_I(\omega) = \frac{A}{\omega Q(\omega)} \tag{36}$$

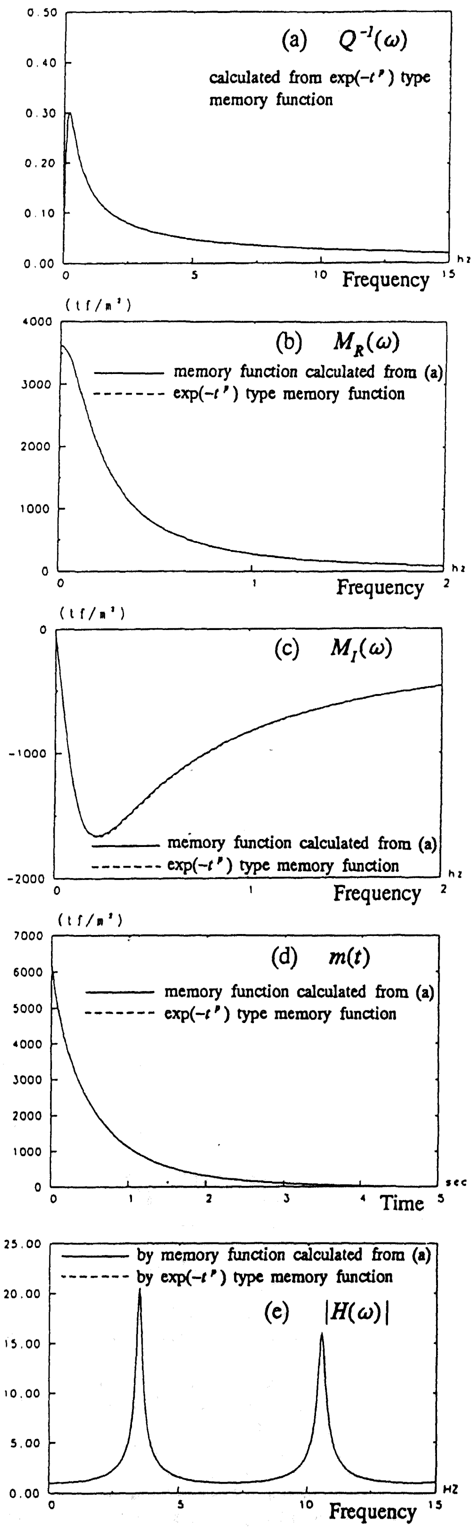

Fig.3 Calculation of exp($-t^p$) type memory function from $Q^{-1}(\omega)$

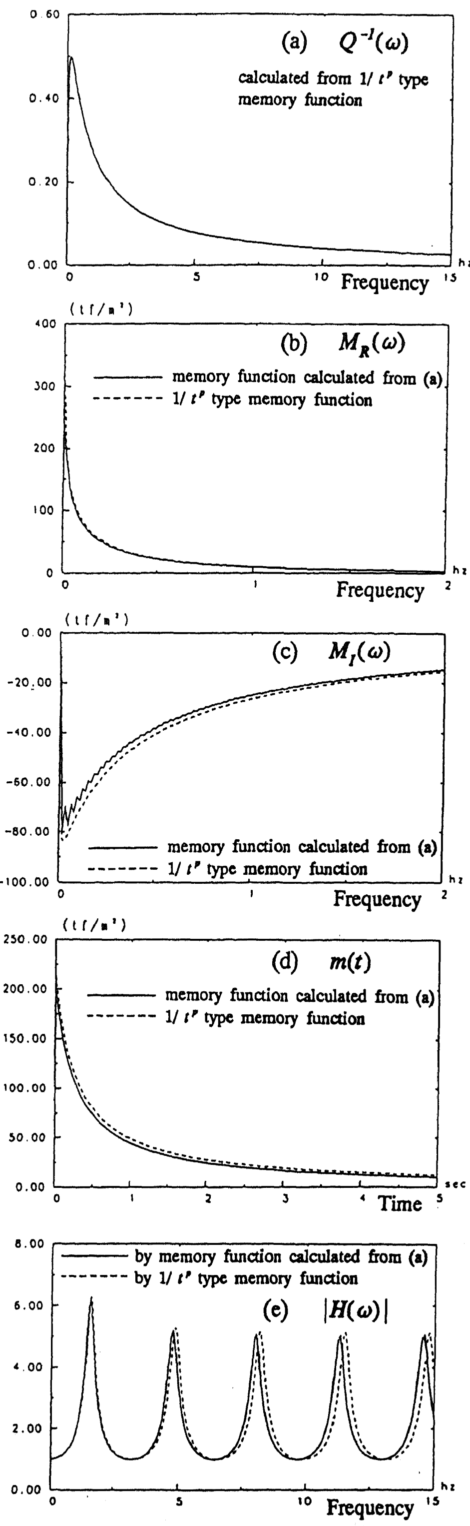

Fig.4 Calculation of $1/t^p$ type memory function from $Q^{-1}(\omega)$

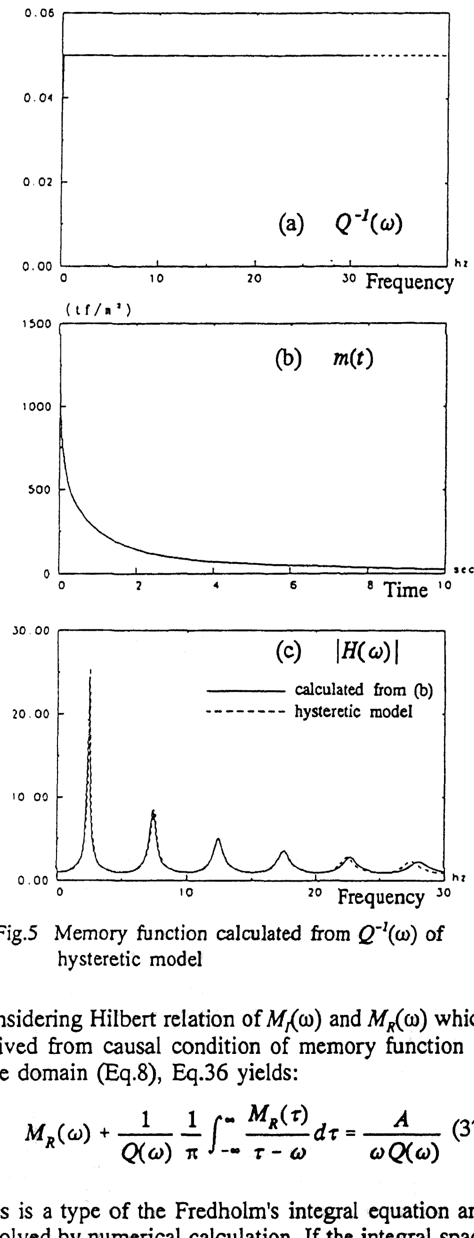

Fig.5 Memory function calculated from $Q^{-1}(\omega)$ of hysteretic model

Considering Hilbert relation of $M_I(\omega)$ and $M_R(\omega)$ which derived from causal condition of memory function in time domain (Eq.8), Eq.36 yields:

$$M_R(\omega) + \frac{1}{Q(\omega)}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{M_R(\tau)}{\tau-\omega}d\tau = \frac{A}{\omega Q(\omega)} \qquad (37)$$

This is a type of the Fredholm's integral equation and is solved by numerical calculation. If the integral space of eq.37 is approximated as finite $[-\omega_N, \omega_N]$,

$$M_R(\omega_j) + \frac{1}{Q(\omega_j)}\frac{1}{\pi}\int_{-\omega_N}^{\omega_N}\frac{M_R(\tau)}{\tau-\omega_j}d\tau = \frac{A}{\omega_j Q(\omega_j)} \qquad (38)$$

where $j = 1\sim N$, $\omega_j = j\Delta\omega$. Applying suitable numerical integration method to Eq.38, following linear equations are obtained:

$$[B]\{M_R(\omega_j)\} = \beta\left\{\frac{1}{Q(\omega_j)}\right\} \qquad (39)$$

where $[B]$ and β are determined by numerical integration method. After solving these equations, $M_I(\omega)$ is calculated from $M_R(\omega)$.

Figs. 3 and 4 present examples of calculation of memory function from Q, for $\exp(-t^{p})$ type and $1/t^{p}$ type. The solid line in (b)~(d) of each figure presents the memory function calculated from Q^{-1} of (a). The dotted line represents the original memory function m(t). They are seen to have nearly same values.

Fig. 5 shows memory function for hysteretic model. Such characteristics are often obtained from the soil data. The Q^{-1} for this model is not fully realized by real and causal memory function, but is approximately calculated numerically in limited frequency range of 0 to 30 Hz in Fig. 5.

7 CONCLUSION

To study wave attenuation in soil, the theory of material with fading memory is introduced. It is shown that the various forms of spring–dashpot model and the creep function model are all special cases of material with fading memory. Some important qualities of attenuation of earthquake wave propagation in viscoelastic material have been investigated by application of memory function. The method of developing the memory function from observed Q is also discussed, for which the solution to the Fredholm's integral equation becomes important. Compared to constitutive models widely used in earthquake engineering, fading memory theory explains wider range of wave attenuation characteristics. If the wave attenuation Q of actual soil is measured, then the actual dynamic behavior of ground can be understood using the memory function based on observed Q.

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