Earthquake Engineering, Tenth World Conference © 1992 Balkema, Rotterdam. ISBN 90 5410 060 5

An invariance problem in probability assignment of recurrence time for next earthquake

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ABSTRACT: This paper considers the conditional probabilites of recurrence times for the next characteristic earthquake in a region and the problem of assigning these probabilities in a self-consistent way from the dataset of observed times between large earthquakes in the region. A consistency principle for these conditional probabilities is stated as a particular of a strong invariance principle. The principle places important invariance requirements on how to model recurrence time probabilities in the frequentist and logical probability frameworks. Examples of two inference methods applied to the data of a region are presented. It is concluded that any inference method to model recurrence times should be self-consistent, that the frequentist method of maximum likehood fails this consistency, and that the relative entropy method with fractile constraints holds the invariance requirement.

1. INTRODUCTION

Structural system design can be optimized by probabilistic methods using risk and reliability estimates and utility functions.In seismic hazard regions, such estimates require the assignment of the joint probabilities for interarrival times and magnitudes of the next earthquakes in lifetime.

For large events in a subduction earthquake area, reliability analysis can be made using simplified models of interarrival times.The simplest of these models deals with the "characteristic earthquakes" in a region, that is the events that repeatidly ruptures the same fault segment and whose magnitudes are near the maximum that the geological fault can originate (see Nishenko and Buland (1987) and Jara and Rosenblueth (1988)).

Other simplifications are the time-predictable earthquake recurrence models developed by Shimazaki and Nakata (1980) and Amagnos and Kiremidjian (1985), in which the interarrival time depends on the magnitude of the preceding event and the rate of stress accumulation along a fault.

This paper only refers to models of marginal probabilities for the interarrival time of the next characteristic earthquake.Reliability analysis in this case also needs the assignment of conditional probabilities of "the recurrence time" for the next earthquake, which is the interval from the elapsed time since the last earthquake to the next event. Otherwise, if the content of information given by the elapsed time from the last earthquake were disregarded, reliabilities could be overestimated.

For systems that involve great risk to life and property, the probability distribution models inferred from data can be justified by rational arguments. Among other rules for inductive inference Jeffreys (1948) stated that a theory of inference must be consistent, i.e. it must not be possible to derive contradictory conclusions from the postulates and any given data. An application of this rule determines the invariance of probability assignments under regular transformations (changes of scale) of random quantities and parameters.

1.1 Strong invariance principle

Jeffreys' consistency rule can be regarded as a general principle that prescribes fundamental requirements for consistency of any inference method, which are also necessary for any posterior action using utility functions. Such a principle has been restated as the following strong invariance principle:

> "Alternate ways of using the same observational data and assumptions in probability assignment should give the same result" (Solana and Lind, (1990)).

A particular case of the strong invariance principle is the principle of invariance of probabilities assigned as alternative ways under conditioning of the domain of the random quantity (see Solana, (1990)).

This consistency principle is here applied to conditional probabilities of recurrence times for the next earthquake in a region.

1.2 Probabilistic interpretations

Most of the distribution models of interarrival times of characteristic earthquakes are essentially frequentist (see Amagnos and Kiremidkjian, (1988)). They are inferred from data using classical statistical methods which assume a given true probability distribution type.

Alternate models of distributions based on logical probabilities can also be considered. Such models correspond to posterior distributions inferred from interarrival time data using either a reference distribution or a prior distribution of parameters.

Principles of invariance can be applied to any inference method for different probability interpretations. Invariance of conditional probabilities of recurrence times is here formulated in Sections 2 and 4 in the cases of frequentist and logical probability frameworks. It places important requirements on how the recurrence times should be modeled in each case. Such invariance requirements are examined in the cases of two inference methods, which have been applied to interarrival time data of large earthquakes given in Utsu (1984) and Nishenko and Buland (1987).

The example presented in Section 3, shows that the Maximum Likelihood Method fails the invariance requirement in the frequentist framework. The Relative Entropy Method with Fractile Constraints (Lind and Solana, (1990)) is applied to the same data in the example presented in Section 4. This method satisfies the invariance requirement in the logical probability framework.

2. THE INVARIANCE PROBLEM IN CLASSICAL PROBABILITY

Invariance of conditional probability estimates of the recurrence time for the next earthquake is formulated in this Section in the classical framework of a frequentist interpretation of probability.

Let $\mathrm{W}_i, i = 1, 2, \cdots, k$, be the values of the random variables $W_i, i = 1, 2, \cdots, k$, of interarrival times of $k+1$ successive earthquakes. As is usual in a frequentist interpretation the random variable W of interarrival time for the next event and the random variables W_i are regarded independent and identically distributed such that

$$Prob(\mathrm{W} \leq w) = Q_W(w) = Q_{W_i}(w) = \\ = Prob(\mathrm{W}_i \leq w), i = 1, \cdots, k. \quad (1)$$

Suppose that the distribution type $Q_W(w)$ is known having the probability density function $q_W(w)$. Consider the sample of size k of the observed interarrival times w_i of random variables W_i. Let $D = \{w_j\}, j = 1, \cdots, k$, be the sample reordered as increasing values of observed data.

In the frequentist probability version an inference method chooses a particular distribution $Q_W(w; D)$ that pertains to the distribution type $Q_W(w)$, in two steps. First, the observed data are encoded in a particular mathematical form, for instance the likelihood or a set of moments, represented by the information $I[D, q_W(w)]$.

Second, the inferred distribution is selected on the basis of the encoded information and the density function $q_W(w)$, as the operation expressed by

$$Q_W(w; D) = e\left[q_W(w), I[D, q_W(w)]\right]. \quad (2)$$

2.1 Alternative conditioning ways

Now, consider the conditional probabilities of recurrence time for the next earthquake. Let w_c be the elapsed time since the last earthquake. These probabilities may be obtained by conditioning the interarrival time values to be on the subdomain $\mathrm{W} > w_c$, as the following two ways:

- i) The estimated probabilities (2) are conditioned using the classical conditional probability rule. The recurrence time distribution in this way is

$$Q_W[(w; D)|\mathrm{W} > w_c] = \frac{Q_W(w; D) - Q_W(w_c; D)}{1 - Q_W(w_c; D)}. \quad (3)$$

- ii) The probability distributions $Q_W(w)$ and $Q_{W_i}(w)$ and the dataset D are initially conditioned by $\mathrm{W} > w_c$. Conditioned distributions so obtained are also independent and identically distributed such that

$$Prob(W \leq w|W > w_c) = Q_W(w|W > w_c) = \\ = Q_{W_i}(w|W > w_c) = Prob(W_i \leq w|W > w_c). \quad (4)$$

Let $q_W(w|W > w_c)$ be the conditional density and let $D_c = \{w_j\}, j = s, s+1, \cdots, k, s \leq 1$, be the resulting censored sample. The recurrence time distribution in this way is selected applying the selection procedure (2) as the operation represented by

$$Q_W[(w|W > w_c); D_c] =$$

$$= e[q_W(w|W > w_c), I(D_c, q_W(w|W > w_c)]. \quad (5)$$

2.2 Invariance requirement

In particular, when the elapsed time w_c equals to any of the interarrival time data $w_j, j = 1, \cdots, k-1$, no new data are introduced through the conditioning procedures into the ways (i) and (ii). In this case, the same data are considered, indeed, in both ways. Therefore the strong invariance principle must be applied as the following invariance statement for consistency of any inference method:

> "The identical conditional probability distribution of recurrence times should be obtained whether the estimated distribution of interarrival times for the given domain is conditioned by an interarrival data-bounded subdomain,or it is directly estimated from the censored interarrival dataset on the same subdomain".

This invariance requirement for conditional distribution of recurrence time may be written by the equivalence of expressions (2) and (5) as follows

$$Q_W[(w; D)|W > w_c] = Q_W[(w|W > w_c); D_c],$$

$$\text{for } w \stackrel{!}{=} w_j, j = 1, 2, \cdots, k-1. \quad (6)$$

3. MAXIMUM LIKEHOOD ESTIMATES

The maximum likelihood estimation (MLE) is probably the most widely applicable frequentist method. It possesses some desirable properties satisfying, for instance, the consistency requirement of invariance under changes of scale.

3.1 Example

In this example, the MLE method is applied to the sample of observed times between large earthquakes in the region of Miyagi–ken–oki, Japan, given in the Appendix. Distributions used for interarrival times in the literature of the exponential type are examined as possible candidates.

The conditional probability distribution of recurrence times have been calculated as the ways (i) and (ii),by using the equations (3) and (5) specialized for MLE estimates,in two cases: (a) When the elapsed time from the last earthquake equals to the lowest interarrival time, (b) When the elapsed time corresponds to July 1992.The results of calculated recurrence times for some probability fractiles have been summarized in the Table 1.

The comparison between recurrence times obtained as the ways i) and ii) shows that they are different in all cases. This example illustrates how the requirement of invariance of conditional probabilities of recurrence times is violated, for instance, in the case (a). Consequently the MLE method fails the invariance principle represented in the frequentist framework by the expression (6), lacking therefore of consistency.

Table 1. MLE Method: Probabilities of recurrence time for the next earthquake in Miyagi-ken-oki region, Japan.

Fractiles	Recurrence time $w - w_c$, in years (i) Conditioning the estimated probabilities	(ii) Estimating the conditional probabilities
(a) Elapsed time : 22.75 years (March 2001)		
(1) Type of distribution: Exponential		
	($\lambda = 0.02576$)	($\lambda = 0.0745$)
0.10	3.81	1.41
0.50	25.17	9.30
0.90	83.29	30.91
0.95	108.37	40.21
(b) Elapsed time : 14.08 years (July 1992)		
(1) Type of distribution: Exponential		
	($\lambda = 0.02576$)	($\lambda = 0.0452$)
0.10	3.81	2.33
0.50	25.17	15.31
0.90	83.29	50.87
0.95	108.37	66.19

4. THE INVARIANCE PROBLEM IN LOGICAL PROBABILITY

Invariance of conditional probability assignments for the recurrence time of the next earthquake is formulated in the logical probability framework (see Carnap (1950)).

Consider W is the value of the random quantity W of interarrival time of the next characteristic earthquake in a region. Let $D = \{w_j\}, j = 1, \cdots, k$, be the set of observed interarrival times in the same region. These observations are assumed exchangeable with themselves and with the unknown value of the next interarrival time.

Logical probabilities are always conditioned by the evidence. These probabilities are the certainty degrees of one or more inferred sentences on a given evidence. Probabilities in this Section correspond to the certainty degrees of the inferred sentence that stands for "the next interarrival time W is lower than w", on the evidence of the sentence that stands for "the observed interarrival time data are $D = \{w_j\}$". Such probabilities are expressed as the conditional distribution

$$Q_W(w|D) = Prob[\mathrm{W} \le w|D] \tag{8}$$

Exchangeability means that the probability distribution (8) is the same for all ways of labelling the observations in the dataset.

In the logical probability version, given a reference distribution function $P_W(w)$ an inference method assigns a conditional probability distribution $Q_W(w|D)$ in two steps. First, the observed data are encoded by a particular information $I(D)$, given for instance as constraints. Second the inferred distribution is obtained on the basis of the encoded information and the reference distribution as the operation symbolically represented by

$$Q_W(w|D) = g[P_W(w), I(D)]. \tag{9}$$

4.1 Alternative ways

Let w_c be the elapsed time since the last earthquake. Conditional probabilities of the recurrence times may be obtained by conditioning the values of interarrival time to be on the subdomain $W > w_c$, as the following two ways:

- i) The assigned probabilities (9) are conditioned by using the multiplicative rule for logical probabilities. The conditional distribution of recurrence time

$$Q_W(w|D \,.\, W > w_c) = \frac{Q_W(w|D) - Q_W(w_c|D)}{1 - Q_W(w_c|D)}. \tag{10}$$

- ii) The reference density $P_W(w)$ and the dataset are conditioned by $W > w_c$. Let $P_W(w|W > w_c)$ be the conditional distribution and let D_c be the resulting censored dataset. The recurrence time distribution is assigned in this way using the same procedure as (9), according to the expression

$$Q_W(w|W > w_c \,.\, D_c) = g[P_W(w|W > w_c), I(D_c)]. \tag{11}$$

The simbol " . " in (10) and (11) means the logic product of the pair of sentences represented by $W > w_c$ and D.

4.2 Invariance requirement

The strong invariance principle can be also applied in this case when the elapsed time w_c equals to any of the interoccurrence time data $w_j, j = 1, \cdots, k-1$. Then, this principle states the following invariance requirement for consistency of any inference method in the logical probability framework:

> "The identical conditional probability distribution for recurrence time is obtained whether the assigned distribution for the next interarrival time is conditioned by any interarrival data–bounded subdomain or it is directly assigned from the censored dataset on the same subdomain".

This invariance requirement for conditional distributions of recurrence times is given by the equivalence of the expressions (10) and (11) as follows

$$Q_W(w|D \,.\, W > w_c) = Q_W(w|W > w_c \,.\, D_c),$$

$$\text{for any } w = w_j, j = 1, 2, \cdots, k-1. \tag{12}$$

5. MINIMUM RELATIVE ENTROPY

The relative entropy method with fractile constraints (REF) is a data–based inference method that employs logical probabilities. It has been recently developed in some papers by Lind and Solana (1988,90).

Among other properties, the REF method satisfies the requirement of invariance under regular transformations of random variables like the classical relative entropy method does (See Shore and Johson (1980)).

The REF method can be described as three distinct steps. First, a set of constraints is encoded from the observed data using the sample rule. This rule assigns the $j/(k+1)$ fractile of the distribution of the next observation for w equals the j–element of the set of observations $D = \ w_j\ $, for any $j = 1, ..., k$.

Second, a set of posterior distributions that satisfy the constraints is chosen, such that each posterior minimizes the relative entropy with regard to the reference distributions of a set of candidates.

Third, the pair of the reference distribution and the assigned posterior that extremizes the relative entropy is selected among the pairs of candidates.

5.1 Example

In this example the REF method is applied to the same interarrival time data as the MLE method in Section 3.1. Likewise the exponential distributions are used as candidate reference distributions.

The posterior conditional distributions for recurrence times have been calculated as the ways i) and ii) by the expressions (10) and (11) specialized for the REF method, in the same two cases a) and b) as remarked in Section 3.1. The results have been summarized in the Table 2.

The comparison between the calculated recurrence times for distinct fractiles shows that the REF method give the same results in the case a), that is, when the elapsed time from the last earthquake equals any of the observed interarrival times. Consequently the REF method satisfies the invariant requirement given by the expression (12) in the logical probability framework.

On the other hand, the recurrence times for each fractile are different in the case b). This result does not imply any violation of an invariance principle. In this case the new datum w_c has been considered, which is differently processed as the ways i) and ii), and the strong invariance principle cannot be invoked. Yet the REF method provides the criterium to select the conditional probabilities in this case. The pair of reference and posterior conditional distributions having minimum relative entropy can be chosen between the calculated pairs of distributions as the ways i) and ii). In particular the selection minimizing the relative entropy in this example corresponds to the value 0.405 remarked in Table 2.

Table 2. REF Method: Probabilities of recurrence time for the next earthquake in Miyagi-ken-oki region, Japan.

Posterior distribution: Fractiles	Recurrence time $w - w_c$, in years (i) Conditioning the assigned probabilities	(ii) Assignment of the conditional probabilities
(a) Elapsed time : 22.75 years (March 2001)		
(1) Type of reference distribution: Exponential		
	$(\lambda = 0.0599)$	$(\lambda = 0.0599)$
0.10	3.5	3.5
0.50	13.58	13.58
0.90	34.75	34.75
0.95	46.32	46.32
	$S_{min} = 0.248$	$S_{min} = 0.248$
(b) Elapsed time : 14.08 years (July 1992)		
(1) Type of reference distribution: Exponential		
	$(\lambda = 0.0378)$	$(\lambda = 0.041)$
0.10	11.24	9.00
0.50	21.07	20.62
0.90	42.85	41.41
0.95	61.03	57.86
	$S_{min} = 0.548$	$S_{min} = 0.405$

6. CONCLUSIONS

1) Models of the recurrence time for the next characteristic earthquake in a region are required for reliability and risk analysis. These models should be inferred on the basis of the available evidential and testable information, such as the observed interarrival times and the information given by the elapsed time from the last earthquake, which is also necessary for updating reliability estimates.

2) If a method for inference of recurrence time models can be applied equivalently in two different ways, it should lead to the same result. This principle for consistency of an inference method has been formalized in the cases of frequentist and logical probabilities as an invariance requirement for assignment of conditional recurrence time distributions.

3) The usual maximum likelihood method fails the invariance requirement in the frequentist case. An example has been presented showing that two different recurrence time probabilities can be equivalently derived from the same data.

4) Only methods that are self-consistent should be applied for assignments of recurrence time probabilities. Fortunately, there is one method namely the relative entropy method with fractile constraints that holds the invariance requirement in the case of logical probabilities. An example applying this method has been presented.

ACKNOWLEDGEMENT

The work reported herein received finantial support from Dirección General de Investigación Científica y Técnica of Spain as a part of the research project PB 89-0077.

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APPENDIX

Table 3. Large Earthquakes of Miyagi Prefecture, Japan (Miyagi-ken-oki region)

Date	Time between eartquakes, years
Sep. 1616	-
June 1646	29.75
Oct. 1678	32.33
Apr. 1736	57.50
May 1770	34.08
Feb. 1793	22.75
July 1835	42.42
Oct. 1861	26.25
Feb. 1897	35.33
Nov. 1936	39.75
June 1978	41.58