

Structural reliability methods for seismic safety assessment

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ABSTRACT: Recent developments in the area of probabilistic methods and structural reliability that are relevant to earthquake engineering problems are reviewed. The paper begins with a discussion of types of uncertainties and methods for their quantification and analysis. The mathematical formulation of the structural reliability problem and existing methods for computation of failure probabilities are then reviewed. Included are methods based on first and second-order approximations, simulation, and various hybrid methods. The topics of reliability sensitivity, updating of reliability, and analysis of reliability uncertainties are discussed. An account of recent developments that combine these probabilistic methods with finite element analysis concludes the paper.

1 INTRODUCTION

Perhaps no other discipline within engineering has to deal with as much uncertainty as the field of earthquake engineering. The randomness in the occurrence of earthquakes in time and space, the vast uncertainty in predicting intensities of resulting ground motions, and our inability to accurately assess capacities of structures under cyclic loading all compel us to make use of probabilistic methods in order to consistently account for the underlying uncertainties and make quantitative assessments of structural safety. While the need for probabilistic methods in earthquake engineering has long been recognized, their use in practice has been limited because of the analytical and computational difficulties that these methods impose on the design and evaluation process. This paper aims at presenting a brief review of recent developments in the field of structural reliability that hopefully will facilitate a wider use of probabilistic methods in earthquake engineering.

The final objective in earthquake engineering often is decision on design specifications. This usually requires risk analysis, involving the assessment of probabilities as well as costs associated with each design alternative and the corresponding consequences, should structural failure occur. A simpler, more practical framework is provided by probabilistic design codes, which implicitly account for the underlying uncertainties and optimize (albeit in an approximate manner) the expected utility derived from each design. The topics of risk analysis and probabilistic codes are not addressed in this paper. However, probabilistic methods that are reviewed are essential elements of any such risk analysis and form the basis upon which probabilistic codes are developed.

Any review on probabilistic methods has to begin with a discussion on the nature of uncertainties and

methods for their modeling and analysis. The Bayesian approach for this purpose is reviewed in the following section. Special emphasis is placed on model uncertainty, an important problem in earthquake engineering which, in the opinion of the writer, has not received adequate attention. In Section 3, general formulations for the structural reliability problem are presented. These include formulations for component and system reliability problems, time-variant reliability problems, and a discussion of reliability measures under conditions of statistical and model uncertainty. Section 4 describes a number of methods for computation of probabilities that are particularly relevant to the structural reliability problem. These include the first- and second-order reliability methods, various simulation methods, the response surface method, and the nested reliability method. Sections 5 and 6 discuss the important topics of probability sensitivity measures and updating of reliability, which are of particular interest in design and planning of systems. The newly emerging field of finite element reliability analysis is briefly reviewed in Section 7, where two numerical examples relevant to earthquake engineering are also presented.

2 MODELING AND ANALYSIS OF UNCERTAINTIES

2.1 Types of Uncertainty

Three types of uncertainty are dominant in structural engineering: (a) inherent randomness, which arises from intrinsic variabilities in materials and in environmental effects, such as loads and support movements; (b) statistical uncertainty, which arises in the course of estimating parameters of probability distributions from observational samples of limited size; and (c) model uncertainty, which arises from the imperfection of

mathematical models used to describe complex physical phenomena, such as models describing loads and capacities of soils or structures. Whereas the uncertainty due to inherent randomness is irreducible, statistical and model uncertainties can be reduced, the former by collection of additional samples and the latter by use of more refined models (see Ditlevsen 1981, 1988; Der Kiureghian 1989). This distinction between the three types of uncertainties is fundamental and deserves careful attention, particularly in connection with developing design codes (Der Kiureghian 1989; Maes 1991). Other sources of uncertainty, such as measurement error and human error, also deserve attention but are not discussed here. Approaches for treatment of these uncertainties may be found in Ditlevsen (1981, 1988), Grigoriu (1984) and Nowak (1986).

A unified approach to the analysis of statistical and model uncertainties is provided by the well known Bayesian updating rule (Box and Tiao 1973)

$$f''(\theta) = c L(\theta) f'(\theta) \quad (1)$$

where $\theta = \{\theta_1, \theta_2, \dots\}$ denotes a vector of model parameters, $f'(\theta)$ is its prior distribution incorporating subjective information, such as an expert's opinion, $L(\theta)$ denotes the likelihood function and incorporates the information gained from observations, $c = \left[\int L(\theta) f(\theta) d\theta \right]^{-1}$ is a normalizing factor, and $f''(\theta)$

denotes the posterior distribution of θ that incorporates both the prior (subjective) information as well as the information gained from the observed (objective) data. For a given observation, the likelihood function is proportional to the conditional probability of making the observation given the parameters. This formula has been used extensively in earthquake engineering for the analysis of statistical uncertainties, but it has seldom been used for model uncertainty analysis. The main difficulties arise in formulating the likelihood function and in evaluating integrals needed to compute the normalizing factor c and the statistics of θ , such as its joint moments and marginal distributions. Examples in the following two subsections demonstrate the use of this formula for statistical and model uncertainty analysis.

2.2 Analysis of Statistical Uncertainty

Let $f(x|\theta)$ denote the joint probability density function (PDF) of x expressed in terms of the unknown parameters θ . If the observation is a set of measured values x_1, x_2, \dots, x_N from a random sampling of x , then the likelihood function is

$$L(\theta) = \prod_{i=1}^N f(x_i|\theta) \quad (2)$$

If the sampling is not random, then one has to use the joint distribution of the sampled values to formulate the likelihood function. In some applications one observes equality or inequality constraints on x instead of directly measuring x . For example, one may observe $h_i(x_i) \leq 0$ for $i=1, 2, \dots, N$, where $h_i(\cdot)$ are constraint functions. One special case is observing bounds on individual elements of x . If the sampling is random,

i.e., x_i are statistically independent, then

$$L(\theta) = \prod_{i=1}^n P[h_i(x_i) \leq 0 | \theta] \quad (3)$$

where $P[\cdot]$ denotes the probability. For the special case of observing bounds, the above probability term is expressed in terms of cumulative distribution functions.

As an example application relevant to earthquake engineering, consider estimation of the mean rate of earthquakes in a region. Let the Poisson model

$$p(n, t) = \frac{(\theta t)^n}{n!} \exp(-\theta t) \quad n=0, 1, 2, \dots \quad (4)$$

describe the occurrences, where $p(n, t)$ denotes the probability of n occurrences in the interval $(0, t)$, and θ denotes the unknown mean rate which is to be estimated. Suppose the available observation indicates x occurrences in the past τ years. Using Eq. 2, the likelihood function then is

$$L(\theta) = \frac{(\theta \tau)^x}{x!} \exp(-\theta \tau) \quad (5)$$

If the prior on θ is gamma distributed with mean μ' and standard deviation σ' (this is the so called conjugate distribution for the Poisson model, see Ang and Tang 1975), the posterior $f''(\theta)$ is also gamma with the mean and variance

$$\mu'' = \frac{\mu'^2 + x\sigma'^2}{\mu' + \tau\sigma'^2} \quad \text{and} \quad \sigma''^2 = \frac{(\mu'^2 + x\sigma'^2)\sigma'^2}{(\mu' + \tau\sigma'^2)} \quad (6)$$

We note that a large prior variance ($\sigma'^2 \rightarrow \infty$) produces $\mu'' = x/\tau$ and $\sigma''^2 = x/\tau^2$, which are the data-based estimates of θ , and a small observation period ($\tau, x \rightarrow 0$) produces $\mu'' = \mu'$ and $\sigma''^2 = \sigma'^2$, which are the prior estimates. For intermediate cases, the posterior mean is somewhere in between the prior and data based means and the posterior variance is usually smaller than either the prior or the data based variance due to the cumulation of information. The posterior distribution of θ reflects our current state of information and uncertainty about θ . Methods for incorporating this kind of uncertainty in reliability analysis are discussed in Section 3.3.

2.3 Analysis of Model Uncertainty

In general, a mathematical model $g(x, \theta) = 0$ describes a relation between a set of observable variables $x = \{x_1, x_2, \dots\}$ through an operator $g(\cdot, \theta)$ with parameters $\theta = \{\theta_1, \theta_2, \dots\}$. The operator might be algebraic, differential or integral in form. Missing variables in the model, incorrect form of the operator, errors in measuring x in the process of model identification, or limited samples to estimate θ are possible sources of model uncertainty. More generally, the model might consist of a set of operators $g_i(x, \theta) = 0, i = 1, 2, \dots$, that describe multiple relations between the variables. The process of formulating the likelihood function depends on the types of model uncertainty present and the form of the model (Bard 1974; Der Kiureghian 1990). In the following, the formulation of the likelihood function for a model used to predict liquefaction in soils is presented.

Motivated by an earlier work of Seed et al. (1985).

Liao et al. (1988) have proposed the following model for the liquefaction of soils under seismic loading:

$$g(N, S, \theta) = \theta_1 N - \theta_2 \ln S + \theta_3 \quad (7)$$

where N represents the standard penetration resistance of soil, S denotes the normalized cyclic stress ratio, and θ_1 , θ_2 and θ_3 are parameters. According to this model, liquefaction occurs when $g(N, S, \theta) \leq 0$ and it does not occur when $g(N, S, \theta) > 0$. The boundary between the two regions, $g(N, S, \theta) = 0$, in the terminology of structural reliability theory, is known as the limit-state surface. It describes the relation between N and S at the onset of liquefaction. Obviously a simple formula such as the above cannot precisely describe as complex a phenomenon as liquefaction. In particular, many variables other than S and N , such as the pore pressure and the soil grain size, may influence the liquefaction potential but are not included in the model. As a result, for any selected θ , one may observe events that violate the above rule. That is, one may predict $g < 0$ where liquefaction does not occur, and $g > 0$ where liquefaction does occur. Data points in the paper by Seed et al. (1985) clearly show such inconsistencies. As a result of this uncertainty, one cannot precisely determine θ , regardless of the amount of data available.

Suppose a set of measurements (N_k, S_k) at $k = 1, \dots, N$ sites is available where it is known whether liquefaction has or has not occurred. Let the true limit-state function for the k -th observation be $g_{true} = g(N, S, \theta) + \gamma_k$, where γ_k is a random correction term that accounts for possible incorrectness of the model form and the effect of all missing variables. If γ_k are assumed to be statistically independent and normally distributed with zero mean (for an unbiased model) and standard deviation σ , the likelihood function for the stated observation is (Der Kiureghian 1990)

$$L(\theta, \sigma) = \prod_{\text{liquefied sites}} \left[1 - \Phi \left(\frac{g(N_k, S_k, \theta)}{\sigma} \right) \right] \times \prod_{\text{non-liquefied sites}} \Phi \left(\frac{g(N_k, S_k, \theta)}{\sigma} \right) \quad (8)$$

where $\Phi(\cdot)$ denotes the standard normal cumulative distribution function. Note that the parameter σ is also considered unknown and is included as an argument of the likelihood function. If γ_k are deemed to be correlated, then the likelihood function involves the multinormal distribution, where the correlation coefficients may also be regarded unknown and included in the argument list of the likelihood function.

The above likelihood function together with a prior distribution can be used in Eq. 1 to derive the posterior distribution of θ and σ . This distribution reflects the uncertainty inherent in the model. The occurrence of liquefaction at a given site can now be predicted probabilistically. Specifically, if the values of N and S for the site are known,

$$P[\text{liquefaction} | N, S] =$$

$$\int_{g(N, S, \theta) + \gamma \leq 0} f''(\theta, \sigma) \phi \left(\frac{\gamma}{\sigma} \right) d\theta d\sigma d\gamma \quad (9)$$

and if N and S are unknown,

$$P[\text{liquefaction}] =$$

$$\int_{g(N, S, \theta) + \gamma \leq 0} f(N, S) f''(\theta, \sigma) \phi \left(\frac{\gamma}{\sigma} \right) d\theta d\sigma d\gamma dN dS \quad (10)$$

where $\phi(\cdot)$ denotes the standard normal density and $f(N, S)$ denotes the joint PDF of N and S .

As described in the following section, the integrals in Eqs. 9 and 10 are typical of many structural reliability problems. Methods for efficient computation of these integrals are described in Section 4. Other integrals are involved in the evaluation of the Bayesian updating rule, such as the one required for computing the normalizing factor c . Numerical methods for computation of such integrals are reviewed by Geyskens et al. (1991). Approaches to incorporating the model uncertainty in reliability analysis are discussed in Section 3.3.

3 FORMULATION OF STRUCTURAL RELIABILITY

3.1 Time-Invariant Problems

The essence of the structural reliability problem is the probability integral

$$P_f = \int_D f(\mathbf{x}) d\mathbf{x} \quad (11)$$

where P_f denotes the "failure" probability, $f(\mathbf{x})$ denotes the PDF of a vector of random variables \mathbf{x} that represent time-invariant uncertain quantities influencing the state of the structure, and D denotes a subset of the outcome space where failure occurs. By "failure," usually the exceedance of a prescribed serviceability or safety limit state is implied. A related measure of interest is the "generalized reliability index", defined by

$$\beta_g = \Phi(1 - P_f) \quad (12)$$

where $\Phi^{-1}(\cdot)$ denotes the inverse of the standard normal cumulative probability. This measure offers certain advantages over the probability measure, particularly in connection with developing codified design rules.

For mathematical analysis, it is necessary to describe the failure domain D in an analytical form. A "component" reliability problem is defined when D is described through a single function, i.e., $D = \{\mathbf{x} | g(\mathbf{x}) \leq 0\}$, where $g(\mathbf{x})$ is known as the limit-state function. The boundary of D is defined by $g(\mathbf{x}) = 0$ and is known as the limit-state surface. These definitions are shown in Fig. 1 for the special case of two variables. A "system" reliability problem is defined when D is described as the union and/or intersection of several subsets. In particular, a "series system" reliability problem is defined by $D = \{\mathbf{x} | \bigcup_i g_i(\mathbf{x}) \leq 0\}$, and a "parallel system" reliability problem is defined by $D = \{\mathbf{x} | \bigcap_i g_i(\mathbf{x}) \leq 0\}$, where $g_i(\mathbf{x})$ denote componental

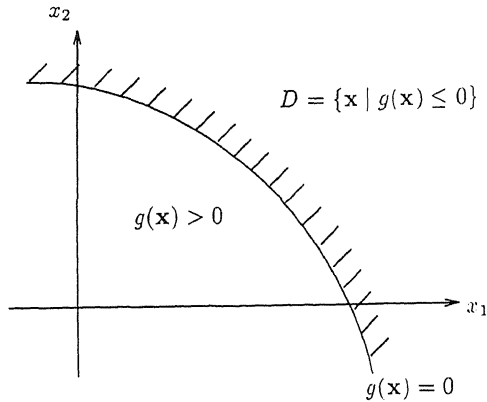


Figure 1. Component reliability problem.

limit-state functions (see Fig. 2). "General system" problems are defined by a cut-set or link-set formulation. The cut-set formulation is defined by $D = \{x \mid \bigcup_k \bigcap_{i \in C_k} g_i(x) \leq 0\}$, where C_k denotes the k -th cut set representing a set of componental limit-states whose joint exceedance constitutes failure of the system. The link-set formulation defines the complement of the fail-

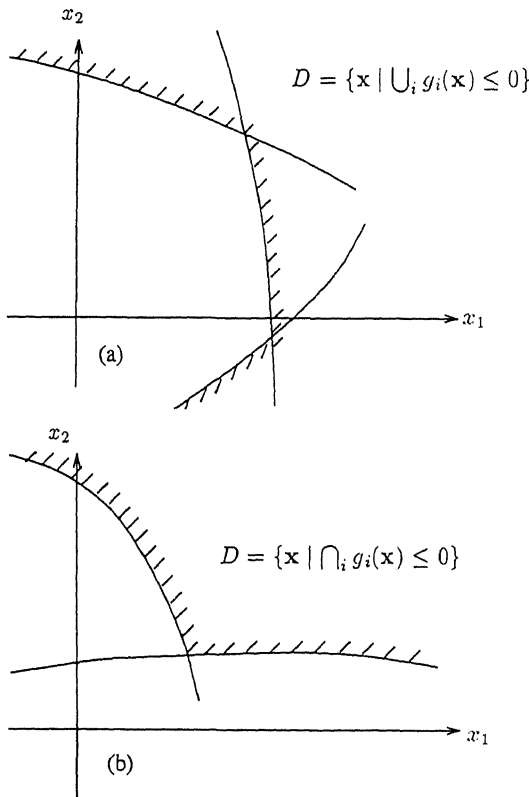


Figure 2. System reliability problems: (a) series system, (b) parallel system.

ure domain as $\bar{D} = \{x \mid \bigcup_k \bigcap_{i \in L_k} g_i(x) > 0\}$, where L_k denotes the k -th link set representing a set of components whose joint survival constitutes survival of the system. Usually, statically determinate structures and elasto-plastic structures with ductile failure modes are modeled as series systems, whereas structures with brittle members require modeling as non-series systems.

The main issue in time-invariant reliability problems is the computation of the probability integral in Eq. 11. Methods available for this purpose are reviewed in Section 4.

3.2 Time-Variant Problems

A reliability problem is said to be time-variant when the limit-state function depends on time, t . One important case is when some of the uncertain variables are stochastic in nature, as in $g(x, y(t))$, where $y(t)$ denotes a vector of stochastic processes. For example, x may denote uncertain mass, stiffness or damping properties of a structure, which are usually time-invariant, and $y(t)$ may denote ground acceleration processes at the support points of the structure. For this class of reliability problems, the failure event constitutes the outcrossing of the vector process $y(t)$ through the limit-state surface $g(x, y) = 0$, as shown in Fig. 3. Usually it is necessary to solve this problem by conditioning on x , i.e.,

$$P_f = \int P \left[\min_{0 \leq t \leq T} g(x, y(t)) \leq 0 \mid x \right] f(x) dx \quad (13)$$

where T denotes the structure lifetime. The conditioned failure probability for given x is solved by the methods of stochastic process theory (Lin 1967), and the integral is evaluated by one of the methods described in Section 4. The nested reliability method described in Subsection 4.3 is particularly suitable for this purpose.

The exact solution of the outcrossing probability unfortunately is not available even for the special case of a linear limit-state function and stationary Gaussian processes. For small failure probabilities, however, the

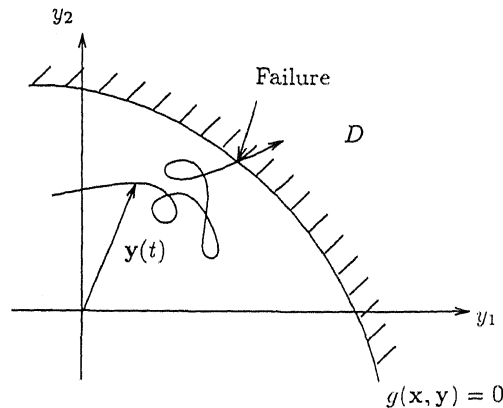


Figure 3. Time-variant reliability problem involving stochastic processes.

following upper bound provides a good approximation (Shinozuka 1964)

$$P \left[\min_{0 < t < T} g(\mathbf{x}, y(t)) \leq 0 \mid \mathbf{x} \right] \leq P \left[g(\mathbf{x}, y(0)) \leq 0 \right] + \int_0^T v(\mathbf{x}, t) dt \quad (14)$$

where $v(\mathbf{x}, t)$ denotes the mean rate of outcrossing of $y(t)$ for given \mathbf{x} . The first term on the right-hand side is a time-invariant reliability problem of the type defined in Eq. 11. The main difficulty is in computing the mean outcrossing rate and considerable effort in recent years has been devoted to finding exact or approximate solutions to this problem. For a linear limit-state surface, the outcrossing problem can be converted to a scalar process upcrossing problem for which a well known solution by Rice (1944) is available. For a non-linear limit-state surface, a generalization of the Rice formula is available that requires integration over the surface (Belyaev 1968). Based on this formula, solutions of the outcrossing rate for special cases of the limit-state surface (e.g., polyhedral or ellipsoid surfaces) have been derived (Veneziano et al. 1977; Ditlevsen 1983, Hohenbichler and Rackwitz 1986b). These solutions are rather involved and require considerable amount of computation. Another idea has been to linearize the limit-state surface at an appropriate point (either the point of maximum mean outcrossing rate or the point of maximum likelihood) and convert the problem into an upcrossing-rate problem (Pearce and Wen 1985). Various choices for the linearization point have been discussed by Ditlevsen (1987). Recently, it has been possible to convert the outcrossing problem into a parallel system sensitivity problem, for which the methods described in Sections 4 and 5 are applicable (Hagen and Tvedt 1991). In spite of these achievements, this area is still in need of research and development before general and effective tools of analysis become available.

Another class of time-variant reliability problems is defined by a limit-state function which is a direct function of time, i.e., of the form $g(\mathbf{x}, t)$. For example, an uncertain structure subjected to a time-varying deterministic load belongs to this class. In some cases it is possible to represent stochastic processes in terms of random variables and deterministic functions of time (Li and Der Kiureghian 1992). In that case, a problem defined by a limit-state function of the form $g(\mathbf{x}, y(t))$ can be converted to one of the form $g(\mathbf{x}, t)$, where \mathbf{x} now includes the random variables defining $y(t)$. The probability of failure for this class of problems is described by the integral in Eq. 11, where the failure domain now is $D = \{ \min_{0 < t < T} g(\mathbf{x}, t) \leq 0 \}$. Hence, methods available for solving time-invariant reliability problems are also applicable to this class of problems. However, caution should be exercised since the limit-state surface in this case is not continuously differentiable and can be strongly nonlinear. An example application later in Section 7 demonstrates this problem.

It is easy to visualize extensions of the above formulations to space-variant reliability problems, such as those involving random fields of material properties or loads, or those where the limit-state function is explicitly dependent on a spatial coordinate. Furthermore, one can formulate problems where the limit-state function is dependent on both time and space coordinates. Methods for solution of such problems are still in early developmental stage.

3.3 Effect of Statistical and Model Uncertainties

The formulation presented in Eq. 11 only accounts for the inherent randomnesses represented by the random variables \mathbf{x} . To explicitly account for statistical and model uncertainties, we write the PDF of \mathbf{x} as $f(\mathbf{x} \mid \boldsymbol{\theta})$ and the componental limit-state functions (disregarding the subscript) as $g(\mathbf{x}, \boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a vector collecting all the uncertain parameters present in the distribution function and the mathematical models describing the limit-states. The probability of failure and the generalized reliability index defined by Eqs. 11 and 12 are now dependent on $\boldsymbol{\theta}$, i.e., $P_f(\boldsymbol{\theta})$ and $\beta_g(\boldsymbol{\theta})$, and hence are uncertain. There are different views as to how these uncertainties should be treated. One view, exercised by Madsen et al. (1986) and Ditlevsen (1982) for example, is to not differentiate between the uncertainties in $\boldsymbol{\theta}$ from those in \mathbf{x} . The result is the expected probability of failure obtained from the total probability formulation

$$E[P_f(\boldsymbol{\theta})] = \int_D f(\mathbf{x} \mid \boldsymbol{\theta}) f''(\boldsymbol{\theta}) d\mathbf{x} d\boldsymbol{\theta} \quad (15)$$

where $f''(\boldsymbol{\theta})$ is the posterior distribution of $\boldsymbol{\theta}$ obtained from Bayesian updating. We note that this integral has the same form as that in Eq. 11, provided \mathbf{x} and $\boldsymbol{\theta}$ are combined to form a single vector. Hence, methods used for the solution of the problem in Eq. 11 can also be used to evaluate the expected probability of failure described above.

Another approach for the analysis of uncertainties recognizes the fundamental difference between the inherent randomnesses represented by \mathbf{x} , which are irreducible, and the statistical and model uncertainties represented by $\boldsymbol{\theta}$, which are reducible. In this case, P_f and β_g are treated as random variables with probability distributions that reflect the effect of statistical and model uncertainties on their estimated values (Der Kiureghian 1989). The cumulative distribution function of β_g , for example, is given by

$$P[\beta_g \leq \beta_o] = \int_{\beta_g(\boldsymbol{\theta}) - \beta_o \leq 0} f''(\boldsymbol{\theta}) d\boldsymbol{\theta} \quad (16)$$

and its PDF can be seen as the sensitivity of the above probability with respect to the parameter β_o . We note that the above integral is also of the type shown in Eq. 11, except that the limit-state function here involves the conditional reliability index $\beta_g(\boldsymbol{\theta})$. The nested reliability method described in Section 4.3 is ideal for solution of this problem.

In application, one is often interested in a simple measure to describe the effect of statistical and model uncertainties on computed P_f and β_g . One such measure is a confidence interval around the computed values with a specified probability (or confidence). The probability distribution described above can be used to generate such intervals. Another convenient measure is the variance of $P_f(\theta)$ or $\beta_g(\theta)$ with θ regarded as random variables. These variances can also be obtained from the distribution described above. However, it is much simpler to use a first-order approximation, which for β_g leads to (Der Kiureghian 1989)

$$\text{Var}[\beta_g(\theta)] \approx \nabla_{\theta} \beta_g \Sigma''_{\theta\theta} \nabla_{\theta} \beta_g^T \quad (17)$$

where $\nabla_{\theta} \beta_g$ is the gradient (sensitivity) vector of β_g at the mean of θ (see Section 5) and $\Sigma''_{\theta\theta}$ is the posterior covariance matrix of θ . This variance together with an assumed distribution can be used to obtain an approximate confidence interval on β_g . A similar formula for the variance of P_f can be written, which is however less accurate due to typically stronger nonlinear dependence of $P_f(\theta)$ on θ . It is better to use the confidence interval of β_g together with Eq. 12 to obtain a confidence interval on P_f .

Existing probabilistic code formats generally do not differentiate between the nature of the uncertainties in \mathbf{x} and θ (Ravindra and Galambos 1978; Ditlevsen 1982). Suggestions for code formats that explicitly differentiate between the two types of uncertainties have recently been made (Der Kiureghian 1989; Maes 1991). These formats tend to penalize for the uncertainty in the computed reliability index and, hence, encourage the use of more data and refined models in engineering design.

4 PROBABILITY COMPUTATION METHODS

A great deal of effort in the past two decades has been devoted to developing efficient algorithms for computing probability integrals of the type in Eq. 11. A straightforward integration, by analytical or numerical means, usually is not possible because of the arbitrary nature of the integration domain and the typically high dimension of the problem. We note that often the size of \mathbf{x} is larger than 10 and it could be as large as several hundred or thousand. Hence, indirect approaches for evaluating the integral are essential. In this section, a review of the most widely used of these methods is presented.

Several of the methods described in the following subsections require transformation of the random variables into the standard normal space. The transformation, which is necessarily nonlinear for non-Gaussian random variables, is expressed as $\mathbf{u} = \mathbf{u}(\mathbf{x})$, where \mathbf{u} has the standard normal density $\phi(\mathbf{u}) = (2\pi)^{-n/2} \exp(-0.5|\mathbf{u}|^2)$, where n denotes the number of random variables. A general form of this transformation is given by Hohenbichler and Rackwitz (1981) and a more convenient form for a particular class of multi-variate distributions is given by Der Kiureghian and Liu (1986). Using this transformation, Eq. 11 is rewritten in the form

$$P_f = \int_D \phi(\mathbf{u}) d\mathbf{u} \quad (18)$$

where D is now defined in the \mathbf{u} space. For notational convenience, we let $G(\mathbf{u}) = g(\mathbf{x}(\mathbf{u}))$ denote the transform of the limit-state function in the \mathbf{u} space.

4.1 First and Second-order Reliability Methods

The first- and second-order reliability methods (FORM and SORM) take advantage of the fact that the point $\mathbf{u}^* = \min(|\mathbf{u}| \mid G(\mathbf{u})=0)$, which is a point located on the limit-state surface with minimum distance from the origin, has the highest probability density among all failure points in the standard normal space. This point is known as the "design point" or the "most likely failure point". It is readily evident that probability densities in the standard normal space are rotationally symmetric and decay exponentially with the square of distance in both radial and tangential directions from \mathbf{u}^* . It follows that the major contribution to the probability integral in Eq. 18 comes from the neighborhood of \mathbf{u}^* , provided the surface is not strongly nonlinear and there is only one significant design point. These conditions are satisfied for most structural component reliability problems. Based on this, the limit-state surface in the neighborhood of the design point is approximated by a first- or second-order surface for which the solution of the probability integral is available. Specifically, in FORM, the limit-state surface is replaced by the tangent hyperplane at \mathbf{u}^* and the first-order approximation of the failure probability is given by

$$P_f \approx P_{f1} = \Phi(-\beta) \quad (19)$$

where $\beta = \alpha \mathbf{u}^*$, known as the (first-order) reliability index, is the distance from the origin and α denotes the unit normal at the design point directed towards the failure set (Fig. 4). In SORM, the limit-state surface is fitted with a second-order surface (usually a paraboloid) at \mathbf{u}^* , and the second-order approximation of the failure probability is given in terms of β and the principal curvatures, κ_i , of the paraboloid. The exact expression of this probability is given in terms of a single-fold inte-

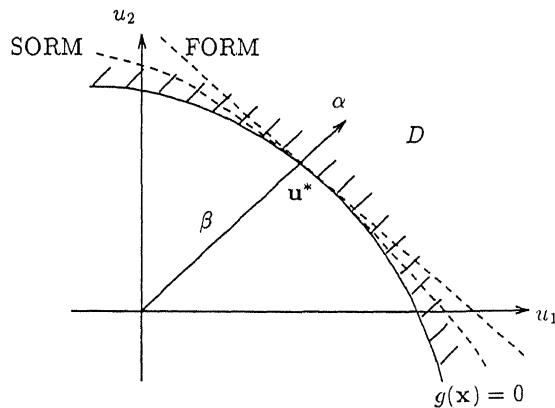


Figure 4. FORM and SORM approximations.

gral, which must be evaluated numerically (Tvedt 1990). A simpler approximation is (Breitung 1984)

$$P_f \approx P_{f2} \approx \Phi(-\beta) \prod_{i=1}^{n-1} (1 + \beta \kappa_i)^{-1/2} \quad (20)$$

Various methods for selecting the approximating paraboloid and computing the principal curvatures are available (Fiessler et al. 1979; Der Kiureghian et al. 1987; Tvedt 1990; Der Kiureghian and DeStefano 1991; Geyskens et al. 1991). An alternative SORM approach has also been suggested by Breitung (1991), where an approximation of the probability by maximizing the log-likelihood in the space of the original variables, \mathbf{x} , is obtained.

An important step in FORM/SORM is the determination of the design point. This requires the solution of a constrained optimization problem. Most algorithms construct a converging sequence of points according to the rule $\mathbf{u}_{k+1} = \mathbf{u}_k + \lambda_k \mathbf{d}_k$, $k=0, 1, \dots$, where \mathbf{d}_k denotes the search direction vector at step k and λ_k denotes the step size. In most efficient algorithms \mathbf{d}_k is expressed in terms of the gradient of the constraint function, ∇G . For example, in an algorithm widely used in reliability analysis,

$$\mathbf{d}_k = \left[\frac{\nabla G}{|\nabla G|} \mathbf{u}_k - \frac{G(\mathbf{u}_k)}{|\nabla G|} \right] \frac{\nabla G^T}{|\nabla G|} + \mathbf{u}_k \quad (21)$$

and λ_k is selected such that $m(\mathbf{u}_{k+1}) < m(\mathbf{u}_k)$, where $m(\cdot)$ is a merit function. A comparative evaluation of this and several other algorithms used in reliability analysis is presented Liu and Der Kiureghian (1990).

When the limit-state surface in the standard normal space is strongly nonlinear, which may arise from strong nonlinearity of $g(\mathbf{x})$ or from strong non-normality of the basic variables \mathbf{x} , multiple design points on the surface may occur with significant neighborhood densities. For such cases, an improved approximation may be obtained by fitting first- or second-order approximations at all significant local design points followed by a series system analysis as described below, provided certain convexity conditions are satisfied. Finding these local design points, however, is a challenge by itself since it is difficult to prevent convergence of the optimization algorithm to the global design point. An algorithm for sequentially finding these points for a special class of reliability problems has recently been developed and an example is presented in Section 7.1.

A general formulation for a first-order system reliability analysis is presented by Hohenbichler and Rackwitz (1983). In this approach, each component limit-state surface is linearized at an appropriate point (the minimum-distance point either on the surface or its intersection with other surfaces) and the probability of system failure is approximated by the probability content in the resulting polyhedral set. It is shown that the first-order approximation for series systems is $P_{f1} = 1 - \Phi_n(\boldsymbol{\beta}, \mathbf{A}\mathbf{A}^T)$, where $\boldsymbol{\beta} = \{\beta_1, \beta_2, \dots\}$ is the vector of component reliability indices, $\mathbf{A} = [\boldsymbol{\alpha}_1 \ \boldsymbol{\alpha}_2 \ \dots]$ is the matrix of unit normals at the linearization points,

and $\Phi_n(\cdot, \mathbf{R})$ denotes the n -dimensional cumulative normal probability with correlation matrix \mathbf{R} . For parallel systems, the approximation is $P_{f1} = \Phi_n(-\boldsymbol{\beta}, \mathbf{A}\mathbf{A}^T)$. Both results require the standard multi-normal probability over polyhedral sets. An approximate method for computing this probability has been developed by Gollwitzer and Rackwitz (1988). Second-order system reliability methods that employ second-order approximations of the component limit-states have also been developed (Madsen et al. 1986; Hohenbichler et al. 1987; Lin and Der Kiureghian 1987).

For series systems, first-order bounds on the system failure probability in terms of individual component (or modal) failure probabilities and cross-modal correlation coefficients (represented by the product $\mathbf{A}\mathbf{A}^T$) have been developed by Ditlevsen (1979). These bounds are easy to compute and essentially require a simple component reliability analysis for each failure mode. These bounds can also be used for computing first-order probabilities for general systems defined in terms of cut sets or link sets (Hohenbichler and Rackwitz 1983).

The major advantage of FORM/SORM is that the probability integral is computed with relatively small number of computations of the limit-state function and its gradient. Typically, for FORM the number of computations of each G and ∇G is of order 10 to 100, and for SORM, depending on the method used, extra computations of G of order $10n$ or n^2 are required to determine the principal curvatures, where n denotes the number of random variables. Note that these numbers are independent of the magnitude of the failure probability. These methods have the additional advantage of providing the coordinates of the most likely failure point and, in SORM, the shape of the limit-state surface in the neighborhood of the design point. Furthermore, as will be described in Section 5, FORM readily provides the sensitivities of the failure probability, which are important in many applications.

4.2 Simulation Methods

A widely used method for computing probabilities is the Monte Carlo simulation method (Rubenstein 1981; Ang and Tang 1984). To apply this method to the structural reliability problem, the integral in Eq. 11 is written in the form

$$P_f = \int I(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \quad (22)$$

where $I(\mathbf{x})$ is an indicator function defined by

$$I(\mathbf{x}) = 1 \quad \text{if } \mathbf{x} \in D \\ = 0 \quad \text{if } \mathbf{x} \notin D \quad (23)$$

Consider a sampling $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_N$ of the random variables according to the density $f(\mathbf{x})$ and let

$$p_i = I(\mathbf{x}_i) f(\mathbf{x}_i), \quad i = 1, \dots, N \quad (24)$$

A mean estimate of the probability of failure, then, is

$$E[\hat{P}_f] = \frac{1}{N} \sum_{i=1}^N p_i \quad (25)$$

where N denotes the sample size. A measure of the error in the estimate is given by the variance

$$\text{Var}[\hat{P}_f] = \frac{1}{N(N-1)} \sum_{i=1}^N (p_i - E[\hat{P}_f])^2 \quad (26)$$

As N approaches infinity, the variance diminishes and the estimate in Eq. 25 approaches the exact failure probability P_f , provided, of course, that the sampling is truly random and representative of the distribution $f(\mathbf{x})$.

Aside from difficulties in producing truly random numbers in accordance with a specified distribution, the main disadvantage of the Monte Carlo simulation method lies in its large computational requirement. Roughly speaking, in order to obtain reasonably accurate probability estimates, it is necessary to have a sample size at least as large as $100/P_f$. In structural reliability problems P_f is typically small, e.g., of order 10^{-2} to 10^{-7} , and the direct Monte Carlo method normally requires tens or hundreds of thousands of repeated computations of the limit-state function. For many engineering applications this is impractical and, hence, attention has been focused on developing more efficient simulation methods. These methods are commonly aimed at reducing the variance of the probability estimate for a given number of simulations, N .

For the structural reliability problem, the most promising variance reduction technique appears to be the importance sampling method (Shinozuka 1983; Harbitz 1986). This method reduces the variance of the estimate by sampling more frequently from inside the failure set. Eqs. 22 and 24 are rewritten in the form

$$P_f = \int I(\mathbf{x}) \frac{f(\mathbf{x})}{h(\mathbf{x})} h(\mathbf{x}) d\mathbf{x} \quad (27)$$

$$p_i = I(\mathbf{x}_i) \frac{f(\mathbf{x}_i)}{h(\mathbf{x}_i)}, \quad i = 1, \dots, N \quad (28)$$

where $h(\mathbf{x})$ is the new sampling density. The ideal choice for $h(\mathbf{x})$ is $I(\mathbf{x})f(\mathbf{x})/P_f$. For this choice, the mean of the estimate is identical to P_f and the variance is zero for any N . However, it is not practical to sample from this ideal density since it would require the same number of g -function computations as the conventional Monte Carlo method. Instead, a sampling density that closely approximates this ideal density but does not require computation of $g(\mathbf{x})$ is employed. Knowledge of the design point(s) from FORM analysis can be particularly helpful in choosing an appropriate sampling density (Harbitz 1986; Schueller and Stix 1987; Melchers 1987). For example, when the safe set is known to be concave, sampling over the half space defined by $\alpha \mathbf{x} \geq \beta$, where β is the reliability index and α is the unit normal at the design point, can be performed. If the safe set is known to be a closed space, then sampling outside of the β -sphere defined by $|\mathbf{x}| \geq \beta$ is a good choice (Harbitz 1986). When the shape of the limit-state surface is not known in advance, then sampling with a normal density centered at each design point is a good choice and has been advocated by a number of investigators (Melchers 1987). Adaptive importance sampling techniques, where the sampling

density is modified during the course of sampling, have also been suggested (Bucher 1988).

It is important to realize that the importance sampling method may converge to a wrong probability estimate if the sampling density is not properly selected. For example, if a zero density is assigned to a subset of D over which $f(\mathbf{x})$ is not zero, then no points will be sampled from that subset and a wrong estimate would be obtained. Hence, it is critical to exercise good judgment in the selection of the sampling density.

Other variance-reduction techniques include the method of antithetic variates, where the variance in the estimate is reduced by creating negative correlation between p_i (Rubenstein 1981), and the method of stratified sampling, and in particular the Latin hypercube sampling, where the number of simulations is reduced by sampling within equally likely intervals of the outcome space (McKay et al. 1979). The latter method, while effective for computing means and variances of functions, is not very effective for computing small probabilities that are typical of reliability problems.

Another class of efficient simulation methods is based on the concept of conditioning. Assume the vector of random variables is partitioned in the form $\mathbf{x} = (\mathbf{y}, \mathbf{z})$ and rewrite Eq. 11 in the total probability form

$$P_f = \int P[g(\mathbf{y}, \mathbf{z}) \leq 0 \mid \mathbf{z}] f(\mathbf{z}) d\mathbf{z} \quad (29)$$

Advantage is achieved if an efficient method is available to compute the conditional probability inside the integral. If this is possible, then simulation can be performed to compute the integral over \mathbf{z} . Note that the integral essentially represents the mean value of the conditional probability with respect to the random variables \mathbf{z} . Hence, the number of required simulations will be small if the conditional probability has a small variance with respect to \mathbf{z} .

One simulation method by conditioning is the so called directional simulation method (Bjerager 1988). The simulation in this method is performed in the standard normal space. The vector of standard variates, \mathbf{u} , is written in the form $\mathbf{u} = R\mathbf{a}$, where $R = |\mathbf{u}|$ denotes the length of the vector and $\mathbf{a} = \mathbf{u}/|\mathbf{u}|$ denotes the unit directional vector. By conditioning on \mathbf{a} , the probability integral is written in the form

$$P_f = \int P[G(R\mathbf{a}) \leq 0 \mid \mathbf{a}] \frac{f(\mathbf{a})}{h(\mathbf{a})} h(\mathbf{a}) d\mathbf{a} \quad (30)$$

where $f(\mathbf{a})$ is the density of \mathbf{a} , which is uniform on the unit sphere centered at the origin, and $h(\mathbf{a})$ is an appropriate sampling density. Since $R^2 = |\mathbf{u}|^2$ has the chi-square distribution with n degrees of freedom, the conditional probability is given by

$$P[G(R\mathbf{a}) \leq 0 \mid \mathbf{a}] = 1 - \chi_n^2(r^2) \quad (31)$$

where r is the root of $G(r\mathbf{a}) = 0$, i.e., is the distance from the origin to the limit-state surface along the direction \mathbf{a} . For a set of simulations $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N$ of \mathbf{a} according to the sampling density $h(\mathbf{a})$, the estimates of the probability of failure are given by Eqs. 25 and 26,

where p_i now is

$$p_i = P [G(R\mathbf{a}_i) \leq 0] \frac{f(\mathbf{a}_i)}{h(\mathbf{a}_i)} \quad (32)$$

Various choices for the sampling density using the knowledge of design point(s) from FORM analysis are suggested by Bjerager (1988) and Melchers (1990).

The directional simulation method is effective when the dimension n of \mathbf{u} is not too large and when the root r_i for each simulated direction \mathbf{a}_i is easily obtained. In some cases, it is convenient to replace the actual surface by an approximating surface to facilitate the root finding process (Lin and Der Kiureghian 1987). Another, non-radial, directional simulation method has been suggested by Hohenbichler and Rackwitz (1988), which is aimed at improving probability estimates based on FORM and SORM.

4.3 Nested Reliability Analysis

Consider the reliability problem defined in the conditional form in Eq. 29. It can be shown (Wen and Chen 1987) that this problem is identical to a reliability problem of the form in Eq. 11 when the limit-state function is taken to be

$$g(u, \mathbf{z}) = u + \beta_g(\mathbf{z}) \quad (33)$$

where $\beta_g(\mathbf{z})$ is the conditional generalized reliability index for given \mathbf{z} and u is a standard normal variate independent of \mathbf{z} . This formulation suggests a nested reliability analysis involving an inner reliability problem for determining $\beta_g(\mathbf{z})$, and an outer reliability problem defined by the above limit-state function and including the random variables \mathbf{z} and u . The inner problem involves the variables \mathbf{y} , which in general can be stochastic processes or random fields.

The above formulation offers an advantage in certain reliability problems when different methods must be used to solve the inner and outer problems. For example, in a time-variant reliability problem involving stochastic processes $\mathbf{y}(t)$, $\beta_g(\mathbf{z})$ may be obtained from random vibration analysis by considering the outcrossing problem associated with $\mathbf{y}(t)$ with \mathbf{z} fixed, whereas the outer reliability problem involving u and \mathbf{z} may be solved by any of the methods described in this section, including FORM and SORM (Wen and Chen 1989; Madsen and Tvedt 1990). This approach has been used for reliability analysis of uncertain systems subjected to stochastic excitation (Igusa and Der Kiureghian 1988). An example in Section 7.1 demonstrates such an application.

4.4 The Response Surface Method

An alternative approach for computing probabilities of the type in Eq. 11 is to replace the integration boundary by an approximating "response" surface and then perform the integration by an appropriate means (e.g., FORM, SORM, simulation) without having to engage the actual limit-state function. This approach is particularly useful when the limit-state function is algorithmic in form (e.g., requires finite element computations) and its gradient, which is necessary in FORM and SORM

analysis, is difficult to compute (Faravelli 1989). To construct the response surface, one typically computes $g(\mathbf{x})$ at a number of points $\mathbf{x} = \mathbf{x}_i$, $i = 1, 2, \dots, N$, and then fits a polynomial surface to the points by the least squares method. If a second-order polynomial is used, calculations at at least $N^2/2 + 3N/2 + 1$ points are necessary, which is a formidable number for large N . To reduce this number the use of polynomial functions excluding cross terms has been suggested (Schueller and Stix 1987). Another suggestion is to reduce the number of random variables by collecting less important variables into a single error term, which is then analyzed by ANOVA (Faravelli 1989). One important consideration in using the response surface methodology is the selection of the "experiment" points, \mathbf{x}_i . There are a number of standard experiment plans for this purpose. For the structural reliability problem, it is clear that these points should be selected in the neighborhood of the design point, which unfortunately is usually unknown in advance.

5 RELIABILITY SENSITIVITY MEASURES

Analysis of probability sensitivities with respect to parameters defining a given problem is an important part of reliability analysis. Parameters, denoted θ , might be those defining the probability distribution function $f(\mathbf{x}|\theta)$, or those defining the limit-state models $g(\mathbf{x}, \theta)$. With respect to a given parameter θ , the sensitivity of the probability of failure is the partial derivative $\partial P_f / \partial \theta$. Collection of these terms in a row vector forms the gradient with respect to θ , $\nabla_\theta P_f = [\partial P_f / \partial \theta_1 \quad \partial P_f / \partial \theta_2 \quad \dots]$. Owing to the definition in Eq. 12, this is related to the gradient of the generalized reliability index by

$$\nabla_\theta P_f = -\phi(\beta_g) \nabla_\theta \beta_g \quad (34)$$

where $\phi(\cdot)$ denotes the standard normal density.

Sensitivity measures are useful in reliability analysis in a number of ways. They provide relative measures of importance of the random variables, are needed in search algorithms for reliability-based optimization or nested reliability analysis (see Section 4.4), are used in reliability uncertainty analysis (Eq. 17), are fundamental to reliability upgrading as discussed in Section 6, and in general are valuable in gaining insight into a problem or in making decisions.

As measures of importance of random variables, it is most useful to compute the gradients with respect to the means, $\boldsymbol{\mu} = \{\mu_1, \mu_2, \dots\}$, and standard deviations, $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \dots\}$, of the random variables. When scaled by the diagonal matrix of standard deviations, these sensitivity vectors, $\text{diag}\{\boldsymbol{\sigma}\} \nabla_{\boldsymbol{\mu}} \beta$ and $\text{diag}\{\boldsymbol{\sigma}\} \nabla_{\boldsymbol{\sigma}} \beta$, represent dimensionless measures of variation in β with respect to equally likely variations in the mean and standard deviation of each variable. The elements within each of these vectors can be compared regardless of the nature of each variable or the units used to represent it. The former vector gives the relative importance of the random variables with respect to their central values, whereas the latter gives the relative importance

with respect to their variabilities. When the variability in a random variable has a relatively small influence on β , then that variable can be replaced with a deterministic quantity, thus reducing the size of the reliability problem. When used in this context, sensitivity vectors are known as omission sensitivity factors (Madsen 1988b).

One important advantage of FORM is that it provides the sensitivity gradients as a byproduct of the algorithm for finding the design point. If θ represents distribution parameters, then (Hohenbichler and Rackwitz 1986a)

$$\nabla_{\theta}\beta = -\frac{\nabla G}{|\nabla G|}\nabla_{\theta}\mathbf{u}(\mathbf{x}^*) \quad (35)$$

and if θ represents parameters in the limit-state function of a component reliability problem, then

$$\nabla_{\theta}\beta = \frac{1}{|\nabla G|}\nabla_{\theta}g(\mathbf{x}^*) \quad (36)$$

where \mathbf{x}^* denotes the design point in the original space and ∇G is the gradient in the standard normal space evaluated at the design point. The latter is usually available from the algorithm used for finding the design point and, hence, one only needs the gradients of $\mathbf{u}(\mathbf{x})$ and $g(\mathbf{x})$ at the design point, which are easy to compute. For system reliability analysis, the derivative of the unit normal at each linearization point is also required. This involves the second derivative of the limit-state function (Bjerager and Krenk 1989). Reliability sensitivity measures for FORM analysis of series and parallel systems are obtained by Bjerager and Krenk (1989) and Madsen (1988a).

Reliability sensitivities can also be obtained by simulation (Ditlevsen and Bjerager 1988; Karamchandani et al. 1988). For this purpose, it is necessary to compute the gradient $\nabla_{\theta}p_i$ for each simulation of p_i in accordance with Eq. 24, 28 or 32. For example, with the importance sampling method, the gradient with respect to the distribution parameters is

$$\nabla_{\theta}p_i = I(x_i)\frac{\nabla_{\theta}f(x_i|\theta)}{h(x_i)}, \quad i=1, \dots, N \quad (37)$$

Note that the gradient computation is necessary only when $I(x_i)=1$, i.e., when the simulated point is inside the failure domain. Unfortunately, derivatives with respect to limit-state parameters cannot be derived with this method because of the non-differential nature of the indicator function $I(\mathbf{x})$. For such parameters, the directional simulation method is applicable, for which (Ditlevsen and Bjerager 1989)

$$\nabla_{\theta}p_i = \frac{2r}{\nabla G \mathbf{a}_i}\nabla_{\theta}G(r\mathbf{a}_i, \theta) \quad (38)$$

The mean and variance estimates of the sensitivities are computed by using the above two derivatives in place of p_i in Eqs. 25 and 26, respectively. One should note, however, that the optimal sampling density for the probability sensitivity in general differs from that which is optimum for the probability itself. Therefore, care must be exercised in the choice of the sampling density if it is used for both the probability and its sensitivity.

6 RELIABILITY UPDATING

Often it is of interest to update the estimated reliability of a structure in light of newfound information. For example, it might be of interest to update the reliability after in situ testing of material strengths. In recent years, this topic has become of particular interest in developing optimum inspection and maintenance strategies for offshore platforms (Madsen 1987; Jiao and Moan 1990).

In general an observation can be one of two basic types: (a) inequality observation, or (b) equality observation. The first occurs when a bound is observed, or the occurrence or non-occurrence of an event is observed. Mathematically, this can be expressed as $h(\mathbf{x})\leq 0$. For example, the observation that the first member of \mathbf{x} is less than 100 can be expressed as $h(x_1)=x_1-100\leq 0$. If an event is observed, such as the initiation of cracks at a point in the structure, then the above formulation applies with $h(\mathbf{x})$ denoting the limit-state function for the initiation of cracks at that location. The equality observation usually is the result of a measurement and it can be mathematically expressed as $h(\mathbf{x})=0$. For example, the observation that x_1+x_2 is measured to be 100 is expressed as $h(x_1)=x_1+x_2-100=0$.

For the inequality-type observation, the updated reliability can be obtained from the conditional probability law

$$P[g(\mathbf{x})\leq 0 | h(\mathbf{x})\leq 0] = \frac{P[g(\mathbf{x})\leq 0 \cap h(\mathbf{x})\leq 0]}{P[h(\mathbf{x})\leq 0]} \quad (39)$$

The numerator on the right-hand side defines a parallel-system reliability problem involving the component limit-states $g(\mathbf{x})$ and $h(\mathbf{x})$, whereas the denominator defines a component reliability problem with the limit-state function $h(\mathbf{x})$. Thus, methods available for component and system reliability analysis are readily applicable to the above updating problem. Generalization of the above formula to the case of several inequality observations $h_i(\mathbf{x})\leq 0$, $i=1, 2, \dots$, should be obvious.

It can be shown (Madsen 1987) that updating with equality observation is equivalent to

$$\begin{aligned} P[g(\mathbf{x})\leq 0 | h(\mathbf{x})=0] &= \frac{P[g(\mathbf{x})\leq 0 \cap h(\mathbf{x})=0]}{P[h(\mathbf{x})=0]} \\ &= \frac{\frac{\partial}{\partial \epsilon} P[g(\mathbf{x})=0 \cap h(\mathbf{x})-\epsilon\leq 0]_{\epsilon=0}}{\frac{\partial}{\partial \epsilon} P[h(\mathbf{x})-\epsilon\leq 0]_{\epsilon=0}} \end{aligned} \quad (40)$$

In this case, the probability sensitivities with respect to the dummy parameter ϵ for the parallel system problem in the numerator and the component problem in the denominator are required. Methods for computation of parallel-system sensitivities are described by Madsen (1988a). Other updating formulations can be found in Jiao and Moan (1990).

7 FINITE-ELEMENT RELIABILITY METHODS

The idea of combining probabilistic methods with finite elements has been around for some time (see Vanmar-

cke et al., 1986, and Der Kiureghian et al., 1991, for recent reviews). Since most structural analysis these days is carried out by finite elements, it is natural that the methods of structural reliability also employ this computational technique. In the context of structural reliability theory described in this paper, the main reason for needing the finite element method is that the limit-state function $g(\mathbf{x})$ is usually an implicit function of \mathbf{x} and can only be evaluated by a numerical algorithm, such as a finite element code. For example, in structural analysis $g(\mathbf{x})$ is often described in terms of load effects such as stresses and deformations, while \mathbf{x} represents elementary input quantities, such as loads and material property constants. Since the relation between load effects and loads and material property constants is usually implicit, the use of a numerical algorithm, such as a finite element code, to evaluate the $g(\mathbf{x})$ function is necessary. Another reason for using the finite element method is to address problems involving random fields of material properties or loads. In this approach, the random field over the domain of interest is discretized and represented by random variables in much the same way as the conventional finite element discretization method. Random-variable reliability methods are then used to compute probabilities of interest. Li and Der Kiureghian (1992) have recently reviewed several methods for discretization of random fields in this context.

An important advantage of combining reliability methods with finite elements is that one gains access to state-of-the-art models of mechanical behavior for reliability analysis. This is particularly important in structural safety studies, where attention naturally is focused on extreme states of a structure and, hence, consideration must be given to grossly nonlinear behavior. A key issue in this regard is the ability to combine the reliability and finite element codes without too much interference in either so that periodic updates in either code can be made. In the experience of the author, the best way is for the reliability code to act as a shell program that runs the finite element code. This approach has been implemented in the CALREL-FEAP finite element reliability code developed by the author and his colleagues at the University of California (see Liu and Der Kiureghian 1991).

Theoretically speaking, any of the reliability computation methods described in Section 4 can be used for finite element reliability analysis. However, there are impediments in straightforward implementation each of these methods. For example, for direct implementation of FORM and SORM, it is necessary to develop efficient and accurate algorithms to compute the response gradients, which typically are not available in existing finite element codes. In recent years notable successes have been achieved in developing such algorithms for both linear and nonlinear structures (Vanmarcke et al. 1986; Santos 1991, Zhang and Der Kiureghian 1991). The conventional simulation method, while appropriate for computing low-order moments of response quantities, usually is prohibitively costly for computing small probabilities that are typical of reliability problems.

More efficient methods, such as directional simulation or simulation by importance sampling, are also difficult to implement. In the directional simulation method, the solution of the root r_i for each simulated direction requires repeated finite element solutions and the number of required computations rapidly grows with the number of random variables. In the importance sampling method, the main issue is in identifying a proper sampling density, which may not be easy if the design point(s) are not known from a previous FORM analysis. The response surface method is easy to implement, since it does not require the response gradients or the selection of a sampling density (Faravelli 1989). However, the challenge here is in selecting informative fitting points (which should be points near the design point) without knowing the actual shape of the limit-state surface. As mentioned earlier, this approach is also restrictive for large number of random variables, since the number of required fitting points (equal to the number of required finite element computations) rapidly grows with the number of random variables. Developments in this field are continuing and improved results and methods should be expected.

In the following subsection we present two examples employing finite element reliability analysis in conjunction with FORM and SORM. These examples are intended to demonstrate the kind of results one may obtain from such analysis. More details on the the data and methods of analysis used can be found in the cited references.

7.1 Example Applications

The first example concerns the first-exursion probability of the horizontal displacement at the top of a three-story, three-bay structure, which has uncertain properties and is subjected to the El Centro 1940 earthquake motion at its base. The girders and columns are modeled by beam elements with elasto-perfectly plastic moment-rotation relations at their ends with the yield moments described as random variables. The problem, hence, is a time-variant reliability problem with a limit-state function of the form

$$g(\mathbf{x}, t) = u_0 - u(\mathbf{x}, t) \quad (41)$$

where u_0 is the allowable displacement threshold, u is the displacement at the top, and the vector \mathbf{x} represents the set of 21 random yield moments at all girder and column ends. As mentioned earlier, the limit-state surface for this type of a problem can be strongly nonlinear with possibly more than one significant design point (see Section 3.2). To solve problems of this type, an active-set gradient projection method has been developed, which is capable of sequentially finding the local design points in the order of their distance from the origin of the standard normal space (Zhang and Der Kiureghian 1992). Application of this algorithm in conjunction with the CALREL-FEAP finite-element reliability program yields two significant design points with first-order reliability indices $\beta_1 = 1.68$ and $\beta_2 = 1.78$. Each design point represents one failure mode of the structure relative to the specified threshold.

The configuration of the structure at each design point is shown in Fig. 5 together with the corresponding time history of the displacement at the top. Solid dots in this figure represent hinges that are in fully plastic state, and open dots represent hinges that have yielded in the past but at the time of failure are in elastic state due to load reversals. These two modes represent the two most likely configurations through which the displacement at the top may exceed the specified threshold. Linearizing the surface at these points and using a series-system reliability analysis for the union of the two failure modes, the generalized reliability index $\beta_g = 1.38$ is obtained, which gives $P_f = 0.083$.

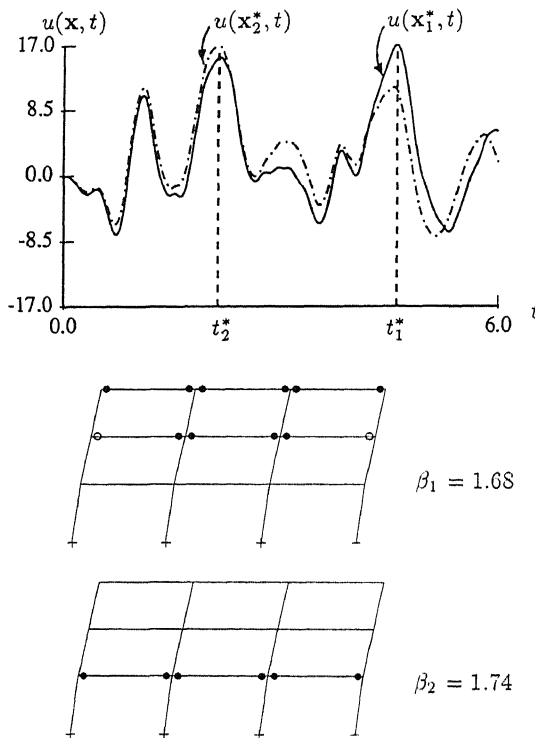


Figure 5. Design point configurations.

The second example, taken from Igusa and Der Kiureghian (1988), examines the influence of uncertainties in structural properties on the response of a tuned primary-secondary system subject to stochastic input. The system, shown in Fig. 6, consists of a 3-degree-of-freedom primary subsystem to which is attached an 8-degree-of-freedom light secondary subsystem. (The primary subsystem has rigid floors and all vertical displacements are ignored.) The mass, damping and stiffness properties of the system are modeled as random variables. The subsystems are designed such that, for the mean values of the properties, the first and third frequencies of the primary are closely tuned to the first and third-fourth frequencies of the secondary, respectively, and hence resonance and amplification of the secondary

response are expected. Attention is focused on the base shear response of the primary and the peak bending moment response of the secondary to an earthquake input modeled as a white noise process. This defines a time-variant reliability problem with a limit-state function of the form

$$g(\mathbf{x}, y(t)) = s_o - s(\mathbf{x}, y(t)) \quad (42)$$

where \mathbf{x} denotes the random system properties, $y(t)$ is the stochastic input, $s(\mathbf{x}, y(t))$ is the response of interest (primary base shear or secondary bending moment), and s_o is the appropriate threshold. For an "acceptable" probability of $P_f = 0.025$, Table 1 lists the normalized required design threshold for each subsystem. Each threshold is normalized with respect to the required threshold in a deterministic system having the mean properties. These results are computed by nested reliability analysis involving random vibrations (inner problem) and SORM (outer problem) analysis. It is evident that the uncertainty in the system properties has a negligible influence on the primary response (only 7% increase in the required design threshold), but a significant influence on the secondary response, i.e., an increase of 45% in the required design threshold. Further sensitivity analysis has shown that, whereas for the primary subsystem the uncertainty in damping is relatively more important, for the secondary subsystem the uncertainties in the mass and stiffness properties of both subsystems is much more important than the uncertainty in damping.

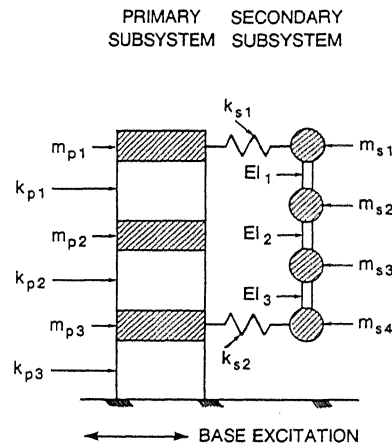


Figure 6. Example primary-secondary system.

Table 1. Normalized thresholds for $P_f = 0.025$

Primary subsystem	1.07
Secondary subsystem	1.45

The above two examples serve to demonstrate useful applications of reliability methods to problems of relevance to earthquake engineering. The domain of application of the methods reviewed in this paper, how-

ever, is much wider than presented by these examples. It is hoped that this paper will serve to stimulate a greater interest in probabilistic methods and their application for improved practice of earthquake engineering.

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