A METHOD TO MODELIZE THE OVERALL STIFFNESS OF A BUILDING IN A STICK MODEL FITTED TO A 3D MODEL

Marc LEBELLE[1]

SUMMARY

The aseismic design of a building using the spectral analysis of a stick model presents very often difficulties in determination of equivalent characteristics of its beams because of the numerous singularities of the structure that make beam theory hypothesis not to be met.

In such cases, it is frequent to adjust these characteristics in order to get for the stick model a behaviour fitted to that of a rough 3D model of the building. This paper presents a straightforward method to treat the results of a static 3D model in order to determine equivalent of characteristics for a general element not limited to the beam formulation.

INTRODUCTION

The aseismic engineering of a building is, in most of civil application based on the spectral analysis of a so-called stick model of the building. For simple buildings the stiffness characteristics of the beams representative of each level of walls between two floors are calculated using the beam theory, making the approximation that cross sections are constant or slowly variable within their length. This hypothesis is generally not met because of the numerous openings in walls that are necessary to the useful exploitation of the building, additions or substraction of walls from one level to another, and other singularity points.

In such cases, one of the frequently used methods to establish stiffness characteristics of the stick model consists in making the design study with a 3D model of the building. This method has the main disadvantage that it does not allow the designer to access easily to the overall forces at each level, which is always a very useful information.

Another method consists in:

- Making a rough 3D model of the building
- Submitting it to unit load cases in each earthquake direction
- Evaluate average displacements and rotations of each level
- Try to approximate by adjustment of the different beam parameters the overall behaviour of the building.

In this second method, the phases of evaluation of average displacements and rotations and adjustment of beam parameters which require iterations are often difficult, are based on the designer judgement and may leave him in deep reflexions about the precision level of what he is doing.

The purpose of the method presented in this paper is to replace these two phases by a straightforward calculation of the stiffness matrices for each level between consecutive floors, that can be used then in the stick model.

HYPOTHESIS

The proposed method is intended to produce data to be input in a stick model to be used in a spectral analysis. It follows that the behaviour of the building is supposed to be linear.

[1] *Société Générale pour les Techniques Nouvelles, Montigny-le-Bretonneux, FRANCE. Email: malebell@club-internet.fr*

The building must also meet a general geometrical criteria which is its global ability to be modelized as a single isostatic line of chained elements elastically or rigidly fixed at a single point at its base. This means that buildings on several separated foundation zones are excluded of the range of applicability of the method. An extension of the method to cases that don't meet this requirement is possible, but the complication of such a method would lead to a huge amount of work to apply it so that it would be economically unacceptable.
The building must also meet the hypothesis that within a level between two floors, the damping ratio of the material constitutive of the building structure is constant. It is probably possible to sophisticate the method in order avoid this necessity, but it has not been done at the present time.

THEORETICAL SUPPORT OF THE METHOD

Notations:

-Upper index EL	: indicates that the quantity refers to an element of a model
-Upper index SM	: indicates that the quantity refers to the stick model of the building
-Upper index 3D	: indicates that the quantity refers to the 3D model of the building
-x	: a displacement of a single degree of freedom of the model
-f	: an force corresponding to a degree of freedom of the model
-w	:a real or virtual work of a force through a displacement
-X	: a displacement vector of the model
-F	: a force vector applied to the model or by an element of the model
-[X]	: a matrix of several displacement vectors of the model
-[F]	: a matrix of several force vectors
-K	: an element stiffness matrix
-[K]	: a global stiffness matrix
-[w]	:an energy matrix for a degree of freedom
-W	: a real or virtual work of a force vector through a displacement vector
-[W]	: an energy matrix for an element or a set of elements
-i,j	: load cases index
-k	: element index
-m	: degree of freedom index
-n	: a level index
-D	: the set of degrees of freedom of the nodes of a level
-n,n+1	: an index for an interval or element between two consecutive levels
-(u,1)	: indicates that the dimension of a column vector previously written is u lines
-(1,v)	: indicates that the dimension of a line vector previously written is v columns
-(u,v)	: indicates that the dimensions of a matrix previously written are u lines and v columns
-[T]	: the geometrical transfer matrix for a force, a degree of freedom, a stiffness matrix from a point to another one.

As a preliminary, we define the notion of energy matrix for several load cases: an energy matrix can be defined for a degree of freedom of an element, an element or a set of elements. It is defined for a set of n load cases acting on a whole structure. For a degree of freedom of an element, its definition is as follows
-if f_i, i=1 to n are the forces for each of the n load cases applied by the element on the external medium along the degree of freedom,
-if x_j, j=1 to n are the displacements of the degree of freedom for the n load cases
-the energy matrix for this degree of freedom and for this set of load cases is $[w]= [w_{ij}]=[1/2f_ix_j]$. It is a square (n,n) matrix. In other terms it is the matrix of real and virtual works of the forces f_i through the displacements x_j.

It is easy to extend the definition to an element by making for the element the sum of the energy matrices for the same set of load cases of all the degree of freedom of the element, and further to a set of elements. It can be demonstrated that in these two last cases, the energy matrix is symmetric non-negative,i.e.:
$[W]^T=[W]$ and $\forall a\neq 0 \Rightarrow a^TWa\geq 0$.

Let us first consider a single two nodes element fixed at one of its ends and free at the other. The stiffness of this system is defined by its stiffness matrix $[K^{EL}]$ (6x6). Now let us suppose $[K^{EL}]$ is unknown and the only information available is the displacement response of this system under any static load applied at the free end. It is possible to rebuild $[K^{EL}]$ from a set of displacements response to 6 independant load cases.

A classical way to do it is to rewrite in a matricial form the classical equations:

$F_i=[K^{EL}]X_i$ i=1 to 6 (1)

becomes:

$[F]=[K^{EL}][X]$ (2)

where $[F]=[F_1, ..., F_6]$ and $[X]=[X_1,.....,X_6]$ are (6x6) matrices.

Under the hypothesis that $F_1,....,F_6$ are independent load cases, this equation can be solved in $[K^{EL}]$ as:

$$[K^{EL}]=[F][X]^{-1} \quad (3)$$

Another way is to consider the energy matrix of this element for the applied loads:

$$[W^{EL}]=1/2[X]^T[F]=1/2[X]^T[K^{EL}][X]=1/2[F]^T[K^{EL}]^{-1}[F] \quad (4)$$

It is possible to compute $[W^{EL}]$ as its first expression $[W^{EL}]=1/2[X]^T[F]$ knowing only the loads applied and the displacement response to these loads.
In order to solve for $[K^{EL}]$, it is possible to inverse either $[W^{EL}]=1/2[X]^T[K^{EL}][X]$ or $[W^{EL}]=1/2[F]^T[K^{EL}]^{-1}[F]$.
For reasons to be explained hereunder, we choose the second solution. The expression for $[K^{EL}]$ is then:

$[K^{EL}]=(2[F]^{T-1}[W^{EL}]^{-1}[F]^{-1})^{-1}=1/2[F][W^{EL}]^{-1}[F]^T$. (5)

The main interest of this formulation in that matter is that even for a more complicated model, the right-hand side member of this equation is composed with terms that can be the result of a summation:
If we suppose the considered element is intended to modelize the part of the 3D model between two consecutive levels, F can be considered as the 6 components force refered to a single point resulting of all the elementary forces that are exerted by this part on the lower and higher parts of the building. And $[W^{EL}]$ can be considered as the sum of all the elementary real and virtual works of all these elementary forces.
This means that if we know from a 3D model the real and virtual works of forces under a set of 6 independent load cases which present a reasonnable repartition in the structure with respect to the real seismic loading, we can build from this information an equivalent stiffness matrix for the considered structure.

THE METHODOLOGY TO MAKE PROFIT OF THE THEORETICAL SUPPORT

From this, the idea of the working method to modelize a building as a stick model fitted to a 3D model is as following:
-submit the 3D model to a set of 6 independent (in terms of resulting forces) load cases F_1 to F_6 applied at the top of the model in order it generates non-zero real works in each interval.
-compute for each interval between two levels the energy matrix for those load cases .
The interval energy matrix is the sum the energy matrices of all the elements located in this interval. As there is no external force applied between two levels, every node located strictly between the two levels is in equilibrium so that the sum of the dof energy matrices for each of these nodes is zero. As a consequence, the computation of the interval energy matrix reduces to the sums of the energy matrices of the dof m, where m covers all the dof of the model at the upper and lower levels of the interval.
-build for each interval the $[W]^{3D}$ matrix the value of which comes from the 3D model and consider this matrix as valid for the stick model as well as the resulting force matrix $[F^{3D]}$ refered to the point where it is intended to put the corresponding node of the stick model.
-compute the fixed 6x6 stiffness matrix of the fixed equivalent element by $[K^{SM}]=1/2\ [F]\ [W^{3D]-1}\ [F]^T$
-complete to the 12x12 stiffness matrix of the-free free equivalent element by using the geometric transfer matrix T associated to the vector from the start to the end of the element:

$$[K^{SM}](\text{free-free})= \left| \begin{array}{c|c} [K^{SM}](\text{fixed}) & [K^{SM}](\text{fixed})[T]^T \\ \hline [T][K^{SM}](\text{fixed}) & [T][K^{SM}](\text{fixed})[T]^T \end{array} \right|$$

-repeat these operations for each level (no need for the first one that is valid for each level if load cases have been applied at the top of the model).

REMARKS AND LIMITATIONS

An interpretation of this method is that it consists to chose for any couple of load cases an average value of each dof that gives the same work (real or virtual) as the 3D model in the same conditions:

$$1/2[F^{SM}_i]^T[X^{SM}_j]=1/2[W^{3D}_{ij}] \qquad \forall\ i,j$$

where $[F^{SM}_i]$ is the resulting force of all elementary forces in the 3D model.

The symmetry of the $[K^{SM}]$ matrix is guaranteed by that of $[W^{SM}]$ guaranteed itself by the virtual works theorem.
The definite positive character of $[K^{SM}]$(fixed) is guaranteed by the fact that it is obtained as real elastic energy terms.
If one calculate the elastic energy for any combination of the 6 independent load cases, say $a_1F_1+a_2F_2+...+a_6F_6$, this energy will be positive and can be expressed as $E(a)=a^T[W]a$ where $a^T=(a_1,a_2,...a_6)$. As a result, $a^T[W]a\geq 0$ for any $a\neq 0$.
The position of the node representing a level is implicitly defined as the point with respect to which is refered [F]. Inversely, if one intend to locate the node representing a level at the center of gravity of masses, he has to refer [F] to this point. But one can also put one node at each level anywhere for stiffness purposes and another at the center of gravity of masses with rigid linking between both. The influence of the real position of the stiff structure elements is automatically taken into account inside the stiffness matrix itself. It is even possible to put only one node at each level and to consider a more complete mass matrix taking account of this shift. From the author's point of view, this is finally the simplest way to proceed because in such a case, all nodes may be located on a single vertical line and so the global forces (particularly moments) above and under a level resulting of the spectral analysis are refered to this same line. But there is no precision criteria in that matter, it's only the designer perception of results that makes the difference.

Note that if the used load cases are the same for each level, the numerical values of F_i are however different because each level as the point with respect to which they are refered is at a different z coordinate.

The load cases applied to the static model have to satisfy some criteria in order to meet the hypothesis of the method:

-no part of the total load has to be applied at a level different of those of the nodes of the stick model. This is because the resulting force acting at both upper and lower levels must be the same.

-If possible, the load cases applied must be as close as possible to the expected global response of the building. But it is also possible, if this is not the case, to eliminate the participation in [W] of those elements that are too much loaded with respect to the latter, with the condition to eliminate it in every component ij of [W].

·-There's no need for the 6 independant load cases to be “ pure ” (for example cases having single components FX, FY, FZ, MX, MY, MZ). The treatment of data is such that the result is not dependent on this hypothesis. It makes it easier to apply load cases including moments in complex geometry because asymetry of the last floor for instance.

·As the repartition of real and virtual works for each i and j is not the same *a priori* in all the elements of the 3D model, the way to attribute a damping ratio to the equivalent element, which classically would have been to make a balance of the different damping ratios of the different elements by their energies is not possible. This is the reason for the hypothesis concerning that matter.

TEST EXAMPLES.

The ability of the method to give good results in terms of the modal analysis of the stick model compared to this of the 3D model is shown in this paragraph through two tests. Compared results are frequencies, effective masses in the three orthogonal main directions and the global deformation.

The 3D structure model is represented in fig. 1 (geometry, restraints and masses locations). It has been made complicated enough so that it is not possible to modelize it with the only beam theory. It has the same geometry in the first and second tests.

The difference between the two tests is only the stiffness of the floors, which is almost infinite in the first test and rather low in the second one.

The aim of the first test is to show that when the floors are “ infinitely ” rigids and the mass are concentrated in the floors, the method gives almost exact results (in this case no energy is spent in the floor elements and the masses are rigidly linked inside a level, so that the behaviour of the 3D model is that of a stick model).

The table 1 gives the comparison of the six first modes frequencies and effective masses for the 3D and the stick models. Fig. 2 gives the visual comparison of four mode shapes of the 3D and stick models. In the stick models, at each level node without mass and rigidly linked to the main node have been added in order to make easier this visual comparison. These nodes do not affect the modal analysis of stick models. One can see the excellent correlation between both model results.

The aim of the second test is to show that when the floors have a finite stiffness, though there is an approximation, the representativity of the model remains quite good and the effect of three different hypothesis concerning the energy of the floors.

In the first hypothesis, the energy of the floor is considered to be spent under the floor (low hypothesis), in the second one half under and half above the floor (medium hypothesis), and in the third one above the floor (high hypothesis).

The table 2 gives the comparison of the six first modes frequencies and effective masses for the 3D and the four stick model obtained. Fig. 3 gives the visual comparison of two mode shapes of the 3D and three stick models. A good correlation between the four models can be observed. It must be noticed that for the second mode the deformed shape of 3D model is opposite to that of the stick models.

Although it is not possible to make a general conclusion about a method which remains only an approximation from such a small amount of tests, it seems like the method works quite well for test 1 and for test 2 there are small differences as expected. In a real case, the designer should examine carefully the importance of the energy matrices of the floors with respect to those of the walls and columns before making an hypothesis concerning the account for this energy.

DISCUSSION.

As shown above, the proposed method has the capability to generate the stiffness matrices for stick models from the results of a 3D model under 6 independant static load cases applied at the top of the building. Of course, this method makes approximations with respect to the 3D model. There is still some parameters on which the designer has to make choices and exert his own judgement:

-The way to apply the forces at the top of the building may have some influence on the results. In particular, for the global Z, MX and MY forces, the results may be quite different for the last level depending on the detail location of corresponding elementary forces:
If they are located on the walls, one will get a high stiffness evaluation and inversely if they are distributed all over the floor, it will be a low stiffness evaluation. In a lower level of a high building, the forces come mainly from the walls and in a higher level, they come mainly from the floor itself, this is the reason why a rough approximation for the six initial load cases is forces applied to the last floor.

-There is no need either that each level stiffness matrix is obtained with the same set of independant load cases, or that in a load case the forces is applied only at the top of the building. It is possible to put in a load case elementary forces at each level but not between two levels.

-The way to use the part of energy that is spent in the floors as stated in §5.

-The capability of the method to represent the behaviour of a building is wider than that by beam elements because for the latter, there are some relationship between the different terms of the stiffness matrix that are imposed. The most general beam element depends on less parameters than the general stiffness matrix. Reciprocally, the perception of a more general behaviour by the designer may be more difficult and except going back to the 3D model, there is no evident way to compute stresses in the structure.

CONCLUSION.

A method has been presented to build the stiffness matrix of the elements of a stick model fitted to the 3D model of a building using the results of 6 independent static load cases applied to this model. The method has proved its capability to give a good approximation of the modal analysis of the 3D model. The limitations and constraints associated to the use of this method have been identified. The advantages of the method are its capability to take into account behaviours more general than those accessible through beams and to be a straightforward procedure. The implementation of tools necessary to apply this method can be made using the standard results of most finite element codes and is not either a difficult task or a great investment so that is is possible for almost any designer to build such tools.

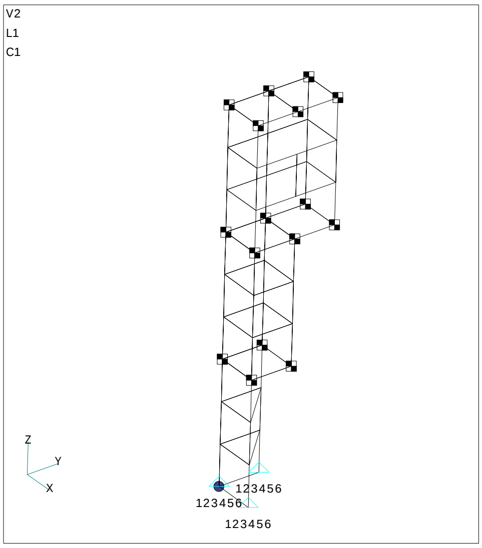
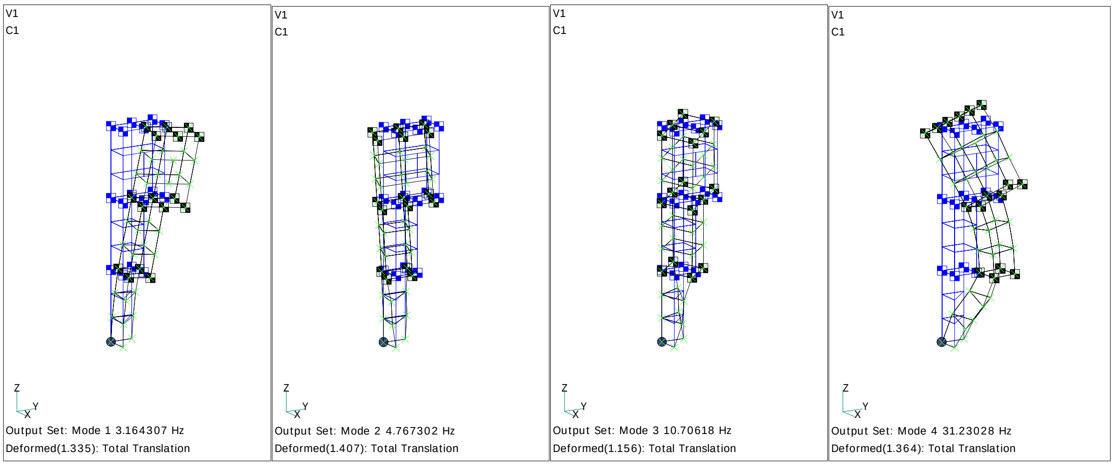


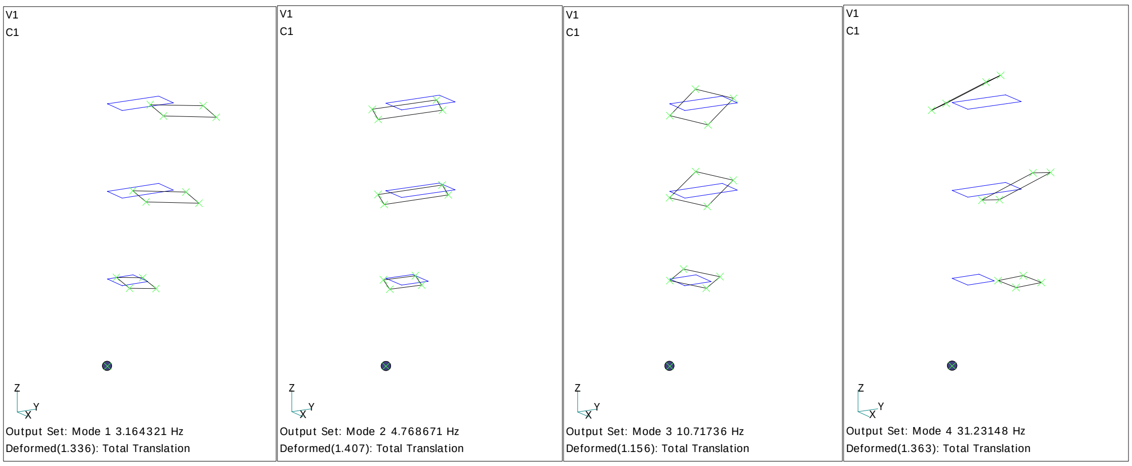
Figure 1.:Geometry, restraints and masses location of the test model

Table 1.: Comparison of frequencies and effective masses for test 1

	3D model					Stick model			
mode number	frequencies	effective masses				frequencies	effective masses		
		x	y	z			x	y	z
1	3.164	36.34766	27.09949	0.82068		3.164	36.33895	27.10843	0.82073
2	4.767	26.85852	35.5318	0.20187		4.769	26.85707	35.51229	0.20194
3	10.706	1.31399	0.67564	0.06892		10.717	1.31254	0.67548	0.06876
4	31.23	6.9693	8.41988	1.93665		31.231	6.96145	8.42769	1.93731
5	38.097	6.14693	6.32702	0.62522		38.028	6.19405	6.34606	0.61477
6	63.07	0.00667	0.36825	58.62837		63.052	0.00809	0.36619	58.45184



3D model

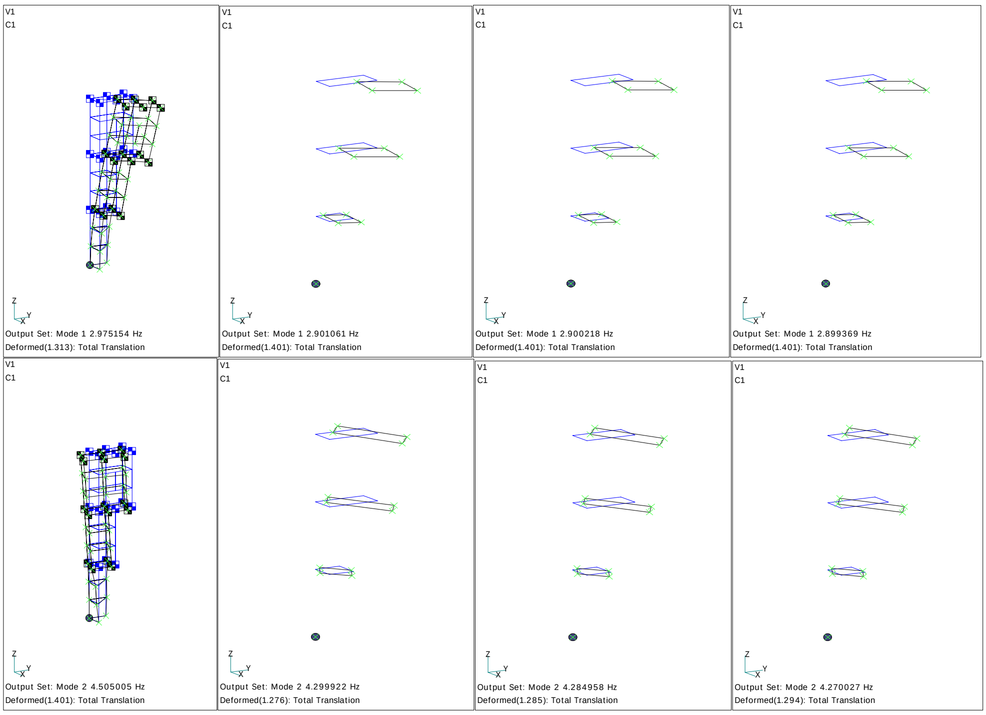


Stick model

Fig 2. Comparison of the mode shapes for test 1

Table 2. Comparison of frequencies and effective masses for test 2.

	3D model					high hypothesis			
mode number	frequencies	effective masses				frequencies	effective masses		
		x	y	z			x	y	z
1	2.975	36.07189	25.2352	0.98801		2.901	34.46863	23.93746	4.77019
2	4.505	24.54774	35.41752	0.17581		4.3	25.80234	34.0211	0.64868
3	10.21	1.70666	1.2025	0.05065		10.645	1.17333	1.13645	0.03703
4	16.646	6.73244	0.80773	7.07961		17.239	0.87552	12.02057	15.83922
5	19.814	2.33204	0.00814	1.17787		18.896	9.70628	0.09221	0.25668
6	23.1	0.14062	11.95577	0.48298		40.69	0.07912	4.0707	3.83753
	medium hypothesis					low hypothesis			
mode number	frequencies	effective masses				frequencies	effective masses		
		x	y	z			x	y	z
1	2.9	34.47932	23.87122	4.79962		2.899	34.48971	23.80475	4.8294
2	4.285	25.78629	33.90366	0.68325		4.27	25.76986	33.78302	0.71907
3	10.594	1.1323	1.30958	0.0844		10.54	1.07713	1.52477	0.17187
4	16.186	1.68298	10.56588	18.60008		15.047	1.87509	8.64387	19.94298
5	18.748	8.71511	0.00069	0.00219		18.588	8.31298	0.00571	0.00084
6	32.199	0.09276	5.66527	0.09387		27.03	0.11791	6.83145	0.01107



3D model **High hypothesis** **Medium hypothesis** **Low hypothesis**

Fig 3. Comparison of the mode shapes for test 2.