0690

TIME INTEGRATION SCHEME THAT ELIMINATES HIGH FREQUENCY NOISE BY DIGITAL FILTER

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SUMMARY

In the dynamic analysis of the nonlinear system with large degrees of freedom, stability of computation is an essential problem. Since noise in high frequency range often triggers instability of the computation, various time integration schemes that have an effect to dampen the noise, such as Wilson method, have been developed.

This paper presents a digital filtering time integration (DFTI) scheme, which eliminates high frequency noise by using digital filtering technique. Usage of digital filter enables the implementation of widely adjustable damping. It can implement various frequency characteristics that other conventional time integration schemes can not have.

Theory and computation process are described in the paper. The frequency characteristics of the DFTI scheme are also investigated and compared with those of conventional time integration methods. It is shown that the DFTI scheme can eliminate high frequency noise efficiently and that it does not deteriorate the accuracy of the analysis in the frequency range of practical concern.

DFTI scheme can work with various conventional time integration methods, and it is applied to a central difference method in this paper. Efficiency of the DFTI scheme is illustrated by the numerical example of a nonlinear dynamic analysis. The results show that the DFTI scheme can eliminate high frequency noise without losing accuracy of the analysis.

INTRODUCTION

High frequency noises often cause serious problem in computation of dynamic system by triggering instability of the computation process, especially when the system has large degree-of-freedom and/or nonlinear material behavior is considered. In such analyses, elimination of high frequency noise is essential.Necessity of the elimination of high frequency has been recognized and various methods have been proposed, such as the β method [Newmark 1959], θ method [Wilson 1968], α method [Hilber et al. 1977], the beta-*m* method [Katona Zienkiewicz 1981], $\theta_1$method [Hoff Pahl 1988a,b], generalized α method [Chung Hulbert 1993]. These methods have algorithmic damping to annihilate high frequency components.

It is also true, however, that larger damping is required to stabilize the computation process in practical cases, and usually physical damping is added. Rayleigh damping, which is given as $C = \alpha M + \beta K$(α and β are parametric constants., M, C and K are mass, damping and stiffness matrices respectively.) is widely used and an $M(M^{-1} K)^m$proportional damping [Munjiza Owen 1998] is also available.

This paper presents a digital filtering time integration (DFTI) scheme. It adds algorithmic damping to the analysis by using digital filtering. Filtering is performed as a part of the time integration process.

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Authors have already presented a time integration scheme with digital filtering [Honda & Sawada 1998], but detail characteristics of the scheme is not discussed. In this paper, we will discuss the frequency characteristics of the DFTI scheme and it is compared with conventional time integration methods; Wilson θ method and Newmark β method.

It is a unique point of the DFTI scheme that it works with various types of conventional time integration methods. Application of the DFTI scheme to the Newmark β method is already shown [Honda & Sawada 1998]. In this paper, we will discuss the way to apply the DFTI scheme to the central difference time integration method.

The first part of this paper describes the theory and the computation process of the DFTI scheme. In the next part, frequency characteristics of the DFTI scheme is described. It is followed by the section that discusses the algorithm to apply the DFTI scheme to the central difference method. Dynamic analysis of a nonlinear system conducted using the DFTI scheme is also introduced and computation results are presented to show the applicability and efficiency of the DFTI scheme.

2 DIGITAL FILTERING TIME INTEGRATION (DFTI) SCHEME

Let us assume $\{x_n\}$ is an original time series and $\{\tilde{x}_n\}$ is a digitally filtered time series. $\{x_n\}$ and $\{\tilde{x}_n\}$ include all variables that appear in the equation of motion, such as displacement, velocity, acceleration and external force. They are referred to as state vectors in the rest part of this paper.

Computation process is summarized as follows. Here we consider a case in which time step is updated from $t = t_{n-1}$ to $t = t_n$. First, we calculate a state vector $\{x_n\}$ by some time integration method. It gives an 'unfiltered' state for the n-th time step. Next, a 'filtered' state vector $\{\tilde{x}_n\}$ is obtained by applying the digital filter as :

$$\tilde{x}_n = \sum_{i=0}^{M} a_i x_{n-i} - \sum_{i=1}^{N} b_i \tilde{x}_{n-i} \tag{1}$$

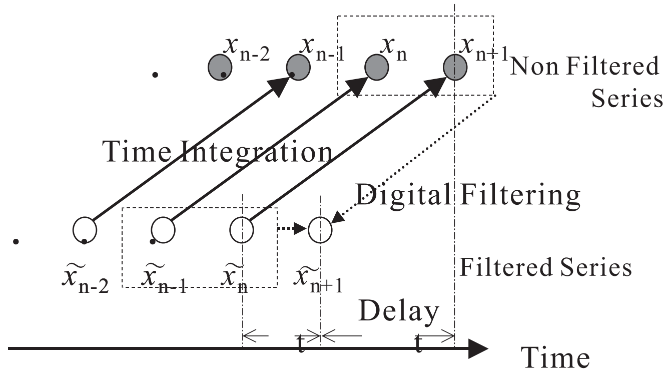
where a_i's and b_i's are coefficients of the digital filter, M and N are number of data at earlier time levels to be used for evaluation of $\tilde{x}_n$.

Digital filter generally has a delay and its effect must be taken into consideration. In order to avert the bad effect of the delay of digital filter, in the DFTI scheme, unfiltered time series $\{x_n\}$ is assumed to have a value at a time level $t = t_n + \tau$ (τ : delay of the filter). This assures that the filtered time series $\{\tilde{x}_n\}$ have the values at time $t = t_n$. Computation process of the DFTI scheme, considering the effect of the delay of the digital filter, is illustrated in **Fig–1**.

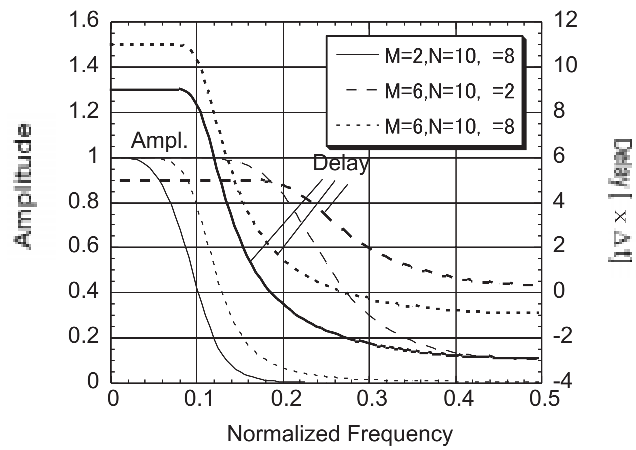
It is also important to consider the effect of frequency characteristics of the digital filter. Gain of the filter should be flatly unit for the low frequency range and it should decrease in the concerned frequency range. Delay of the filter should also take constant value in the concerned frequency range.

Here we use the rational Legendre filter [Sato, 1976]. It is a flat-delay flat-attenuation IIR (infinite impulse response) filter. Coefficients (a_i's and b_i's) of the rational Legendre filter are given as a function of M, N (see eqn.(1)) and delay τ, and therefore they can be easily calculated. Maximally flat delay (MFD) filter [Thiran, 1971], which has much larger attenuation in the relatively low frequency range and therefore it serves as a more severe filter, is also available.

Examples of the attenuation and delay of the rational Legendre filter are shown in **Fig.–2**.



Figure–1 Computation process of DFTI



Figure–2 Frequency characteristics of the rational Legendre filter : Attenuation and delay take flat value in the low frequency range.

3 FREQUENCY CHARACTERISTICS OF DFTI SCHEME

In this section, frequency characteristics of the DFTI scheme are investigated and compared with those of the Wilson θ and the Newmark β method.

Frequency characteristics is investigated by the behavior of the solution of the equation of motion with the natural period of ω :

$$\ddot{x} = -\omega^2 x. \tag{2}$$

Usually the DFTI scheme is used with a conventional time integration method. In order to consider the effect of DFTI, however, we use the DFTI scheme with the analytical solution of equation (2). When state vector at time $t = t_n$ is given as the initial condition, analytical solution at $t = t_n + \Delta t$ is given as,

$$\boldsymbol{x}(t + \Delta t) = \boldsymbol{T}\boldsymbol{x}(t) \tag{3}$$

where

$$\boldsymbol{x}(t) = \left\{ \begin{array}{c} x(t) \\ \dot{x}(t) \end{array} \right\}, \qquad \boldsymbol{T} = \left[\begin{array}{cc} \cos(\omega\Delta t) & \frac{1}{\omega}\sin(\omega\Delta t) \\ -\omega\sin(\omega\Delta t) & \cos(\omega\Delta t) \end{array} \right] \tag{4}$$

By assuming a vector as,

$$\boldsymbol{x}(t)=\{\tilde{x}_n,\dot{\tilde{x}}_n,\tilde{x}_{n-1},\dot{\tilde{x}}_{n-1},\cdots,\tilde{x}_{n-N+1},\dot{\tilde{x}}_{n-N+1},x_n,\dot{x}_n,x_{n-1},\dot{x}_{n-1},\cdots,x_{n-M+1},\dot{x}_{n-M+1}\}^{\mathrm{T}}, \tag{5}$$

free oscillation solution of the equation (2) is written, assuming matrix $\boldsymbol{A}$ appropriately, as,

$$\boldsymbol{x}_{n+1}=\boldsymbol{A}\boldsymbol{x}_n. \tag{6}$$

Eigen values of $\boldsymbol{A}$, $\lambda_i(\omega)$, indicates the estimated period T_i' and algorithmic damping $\bar{\xi}_i$ as,

$$T_i' = \frac{2\pi}{\bar{\omega}_i} \tag{7}$$
$$\bar{\omega}_i = \bar{\Omega}_i/\Delta t \tag{8}$$
$$\bar{\xi}_i = -\mathrm{Re}(\log(\lambda_i)/\bar{\Omega} \tag{9}$$
$$\bar{\Omega}_i = \mathrm{Im}(\log(\lambda_i)) \tag{10}$$

Frequency characteristics of the Newmark β method and the Wilson θ method are also estimated in the same manner.

Comparison of frequency characteristics of the DFTI scheme with the rational Legendre filter ($N=6, M=10, \tau=6$), the Newmark β method ($\beta=\frac{1}{4}$) and the Wilson θ ($\theta=1.4$) is shown in **Fig.–3**. In **Fig.–3**(a) the estimation error of the period $\frac{T'-T}{T}$ is plotted against the normalized frequency, and it is shown that the error of the DFTI scheme is smaller than those of other two methods. For example at the frequency $\frac{\Delta t}{T}=0.1$, the DFTI scheme takes about zero while other methods take values of 0.02 or larger. This indicates that the DFTI scheme does not deteriorate the computational accuracy for a relatively wide frequency range. **Fig.–3**(b) shows that damping of DFTI is smaller in the low frequency range and larger in the high frequency range, than those of the Wilson θ method. The Newmark β method does not have algorithmic damping when $\beta=\frac{1}{4}$. This result indicates that DFTI scheme can eliminate high frequency component efficiently and adds little damping to the relatively low, and practically important, frequency range.

4 APPLICATION TO CENTRAL DIFFERENCE METHOD

In this section, application of the DFTI scheme to the conventional time integration method is discussed, taking the central difference method as an example.

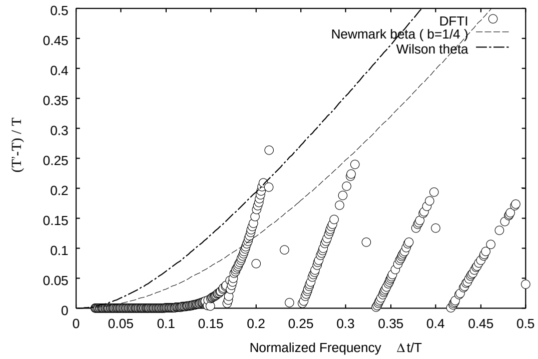
Let us consider the equation of motion as,

$$[M]\ddot{x}+[C]\dot{x}+[K]x=p(t) \tag{11}$$

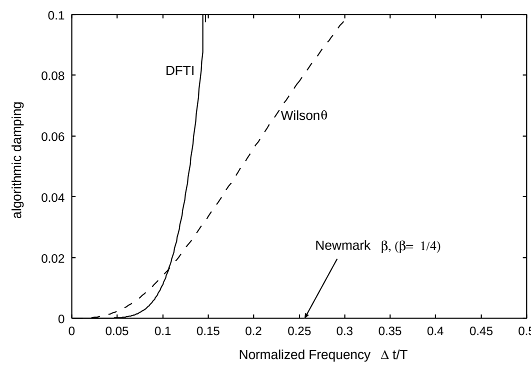
where $x, p, [M], [C], [K]$ denotes displacement, external force vector, mass, damping and stiffness matrices. The central difference method rewrites this equation as,

$$[M]\frac{x_{n[+1]}-2x_n+x_{n[-1]}}{\Delta t^2}+[C]\frac{x_{n[+1]}-x_{n[-1]}}{2\Delta t}+[K]x_n=p_n. \tag{12}$$

In equation (12), displacement at time level $t=t_{n+1}$ and $t=t_{n-1}$ are expressed as $x_{n[+1]}$ and $x_{n[-1]}$, instead of x_{n+1} and x_{n-1} as they are in the ordinary formulation. The DFTI scheme first computes the



(a)Relationship of the accuracy of estimated period and the normalized frequency.



(b) Relationship of the algorithmic damping and the normalized frequency.

Figure–3 Frequency characteristics of the accuracy of estimated natural period and algorithmic damping. They are compared among DFTI scheme, Newmark β method and Wilson θ method. Frequency is normalized by multiplying Δt.

unfiltered state vector $\{x_n\}$ and then estimates the filtered state vector $\{\tilde{x}_n\}$ by digital filtering. Since x_{n+1} and $x_{n[+1]}$ in equation (12) are not always identical, we need to assume a state vector $x_{n[+1]}$ that satisfies equation (12). Considering this, we adopted $x_{n[+1]}$, which is used as the state vector at time $t = t_{n+1}$ and it is used only when the equation of motion at time $t = t_n$ is considered. $x_{n[+1]}$ is used to satisfy the equation of motion at time $t = t_n$ and not at time $t = t_{n+1}$. Therefore $x_{n[+1]}$ and $x_{n[-1]}$ are included in the state vector $\{x_n\}$ for the time level $t = t_n$.

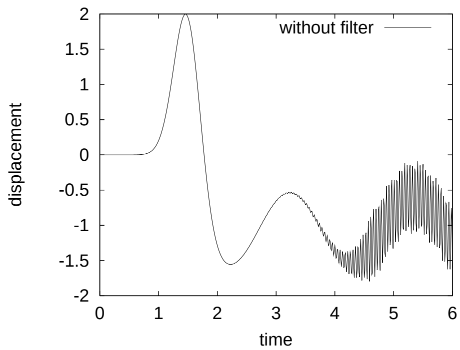
The computation process to update from time level $t = t_n$ to $t = t_{n+1}$ can be summarized as follows. First, we estimate $\tilde{x}_{n[+2]}$ from $\tilde{x}_{n[+1]}$ and $\tilde{x}_n$ by using equation (12). By repeating the same procedure $\tau+1$ times, we obtain $\tilde{x}_{n[+\tau+2]}$, $\tilde{x}_{n[+\tau+1]}$ and $\tilde{x}_{n[+\tau]}$. They corresponds to the unfiltered state vectors $x_{n+1[+1]}$, x_{n+1} and $x_{n+1[-1]}$. Digital filtering of these vectors give the filtered state vectors $\tilde{x}_{n+1[+1]}$, $\tilde{x}_{n+1}$ and $\tilde{x}_{n+1[-1]}$. It should be noticed that both filtered and unfiltered state vectors satisfy the discretized equation of motion (12) at every time level.

5 NUMERICAL EXAMPLE

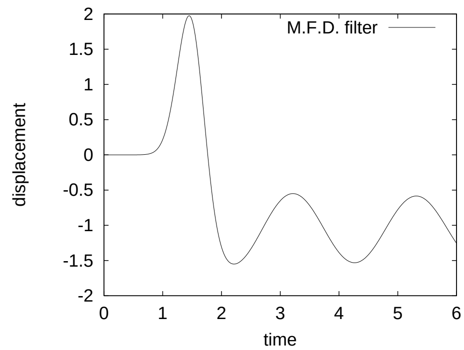
DFTI scheme is applied to the dynamic analysis of a nonlinear 2-degree-of-freedom system. The system consists of two masses and two springs. Bi-linear model is assumed as the restoring force-displacement relationship of the springs.

Natural periods of the first and second modes of the system are 3.74 and 0.05 seconds. In order to make the system unstable, Rayleigh damping is added so that damping ratio becomes negative in the high frequency range. Damping ratio is set 0.01 and -0.01 for the first and second modes' natural frequencies, respectively. Because of the negative damping (-0.01) added to the high frequency range, the system has a tendency to diverge. In the analysis, Ricker wave is used as the input motion.

Time histories of the displacement of mass at the top are shown in **Fig–4**. Due to the negative damping added to the system, analysis without filtering has a strong trend of divergence, while the analysis by the DFTI scheme is stable.



(a) case without filtering.



(b) case with the DFTI scheme.

Figure–4 Time histories of the displacement obtained by the dynamic analysis of the nonlinear 2-DOF system. The diverging tendency observed in the case without filtering is constrained in the analysis conducted with the DFTI scheme.

6 CONCLUSION

Digital filtering time integration (DFTI) scheme, which eliminates the high frequency noise of dynamic analysis, is presented and its theory and computation process is discussed.

The DFTI can be used with various time integration scheme. In this paper, algorithm to apply it to the central difference method is presented. Frequency characteristics of the DFTI scheme is also discussed and compared with those of the Wilson θ method and the Newmark β method. Comparison results show that the DFTI scheme can implement a frequency characteristics that is suitable for the purpose of numerical computation.

Dynamic analysis of the nonlinear system with strong trend to diverge is also conducted. The analysis which diverged when it was conducted with the ordinary method was stably analyzed by using the DFTI scheme. The computation results can be considered to show the applicability and efficiency of the presented DFTI scheme.

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