STRUCTURAL DYNAMIC RESPONSE MITIGATION BY A CONTROL LAW DESIGNED IN THE PLASTIC DOMAIN

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SUMMARY

In previous papers [Baratta and Corbi, 1999] a control algorithm has been proposed, referred to an elasto-plastic structural model and based on the minimisation of a functional of the elasto-plastic response, constrained by an upper bound on the control energy; the functional has proved to be a reliable index of the control effectiveness in the plastic domain, while the assumed algorithm, tested for variously shaped dynamic loads, has shown to achieve an extremely reduced structural response when compared to the uncontrolled one.

In the paper one considers the possibility to relate the results deriving by the adoption of such a procedure to those one can get by following an “exact” control strategy which minimises, for the same assumed forcing function, the maximum value of the plastic displacement while containing the maximum control force under the highest value it reaches in the “plastic-controlled” structural system numerical simulation. Furthermore, a comparison is proposed with the response of the elasto-plastic model controlled by means of a classical elastic norm strategy. The conclusion is that, whereas the plastic component is considerable, the deepest mitigation of the structural response is best accomplished by means of the “plastic- control” procedure.

INTRODUCTION

The problem to mitigate the response of a structure subject to a seismic action can be approached by the implementation of active control techniques. In seismic engineering, due to the possibility that the quake intensity exceeds the design peak acceleration, even if an effective control action is applied, an high probability exists that the structural response exceeds the elastic threshold. For the elastic-plastic model assumed for the considered s.d.o.f. structure, a control strategy has been developed [Baratta and Corbi, 1999b] in order to attenuate the plastic response of the structure, by following the usual technique for bounding displacements in elastic plastic dynamics [see i.e. Maier, 1973 and Capurso, 1975]. The approximate technique is pretty reliable and particularly desirable whereas large plastic response components are supposed to occur.

THE STRUCTURAL MODEL AND THE EQUATIONS OF THE MOTION

Let consider the equations of the dynamic equilibrium governing the motion of a s.d.o.f. structural system [Figure 1] with damping and frequency respectively ζ ed ω_o, subject to a forcing function *f*(*t*) and controlled by a force *c*(*t*),

linearly dependent on the displacement variable *u*(*t*), splittable in an elastic $u_e(t)$ and a purely plastic $u_p(t)$ component in such a manner that

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$$u(t) = u_e(t) + u_p(t) \tag{1}$$

And on its derivative $\dot{u}(t)$ by means of the control parameters μ e ϖ. In the elastic field one can write

$$\begin{cases} \ddot{u}_E(t) + 2\zeta\omega_o\dot{u}_E(t) + \omega_o^2 u_E(t) + c_E(t|\mu,\varpi) = f(t) \quad c_E(t|\mu,\varpi) = 2\mu\varpi\dot{u}_E(t) + \varpi^2 u_E(t) \\ u_E(0) = u_o \quad ; \quad \dot{u}_E(0) = \dot{u}_o \end{cases} \tag{2}$$

where $u_E(t)$ is the elastic response to *f*(*t*), while, whether the excitation produces any excursion in the plastic field, the standard equation of the motion is expressed by

$$\begin{cases} \ddot{u}(t) + 2\zeta\omega_o\dot{u}(t) + \omega_o^2 U(u,\dot{u},u_p,\dot{u}_p) + c(t|\mu,\varpi) = f(t) \qquad c(t|\mu,\varpi) = 2\mu\varpi\dot{u}(t) + \varpi^2 u(t) \\ \dot{u}_p(t) = H[u(t) - u'_o - u_p(t)]\cdot\dot{u}(t) + H[u''_o + u_p(t) - u(t)]\cdot\dot{u}(t) \end{cases} \tag{3}$$

where the terms $U(t), u_p(t), \dot{u}_p(t)$ are given in Table 1, *H*(*x*) is the step function and

$$u'(t) = u'_o + u_p(t) \quad ; \quad u''(t) = u''_o + u_p(t) \tag{4}$$

$$\bar{u}_p(t) = u'(t) - u'_o = u''(t) - u''_o = \frac{u'(t) - u'_o + u''(t) - u''_o}{2} \tag{5}$$

$$u'_o = s'_o/\omega_o^2; \qquad u''_o = s''_o/\omega_o^2; \qquad (s'_o, u'_o > 0; s''_o, u''_o < 0) \tag{6}$$

with s'_o, s''_o the values at the yielding threshold of the variable

$$s(t) = \omega_o^2 u_e(t) = \omega_o^2[u(t) - u_p(t)] \tag{7}$$

which represents the *standard* (that' s to say with the physical dimensions of an acceleration) shear stress in the piles.

Table 1: The $U(t), u_p(t), \dot{u}_p(t)$ values in the elastic and plastic field.

Elastic field:	**Plastic field:**	**Plastic field:**
$u''(t) \le u(t) \le u'(t)$	$\dot{u}(t) \ge 0 \quad ; \quad u(t) \ge u'(t)$	$\dot{u}(t) \le 0 \quad ; \quad u(t) \le u''(t)$
$U = u(t) - \bar{u}_p$ $u_p(t) = \bar{u}_p$ $\dot{u}_p(t) = 0$	$U = u'_o$ $u_p(t) = u(t) - u'_o$ $\dot{u}_p(t) = \dot{u}(t)$	$U = u''_o$ $u_p(t) = u(t) - u''_o$ $\dot{u}_p(t) = \dot{u}(t)$

Moreover

$$s(t)\dot{u}_p(t) = \omega_o^2[u(t) - u_p(t)]\dot{u}_p(t) \ge 0 \tag{8}$$

In the elasto-plastic field, marking by $\bar{s}(t)$ an admissible law $(s'_o \le \bar{s}(t) \le s''_o)$ of the standard shear stress, the following relations are met as well

$$[s(t) - \bar{s}(t)]\dot{u}_p(t) \ge 0 \tag{9}$$

$$[s'_o - \bar{s}(t)][s''_o - \bar{s}(t)] \le 0 \tag{10}$$

$$\left[s_o' - \bar{s}(t)\right]\left[s_o'' - \bar{s}(t)\right]\dot{u}_p(t) = 0 \tag{11}$$

that’s to say, from eqn (9) to eqn (11), the Drucker’ s postulate, the yielding condition and the rule of plastic flow.

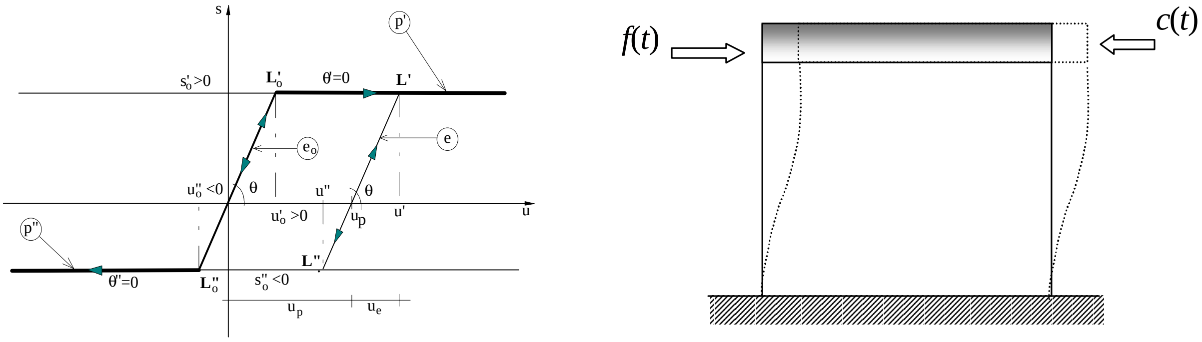

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Figure1: The considered elastic perfectly plastic model a) of the s.d.o.f. system b), with $\theta = \omega_o^2$.

EVALUATION OF THE PLASTIC DISPLACEMENT IN A CONTROLLED SYSTEM.

In order to evaluate the displacement induced in the structural system during the seismic motion, by following the response bounding approach by Maier [Maier, 1973], let consider**:**

1. The linear elastic response $u_E(t), s_E(t)$ to the same seismic excitation with the same initial conditions, which satisfies eqn (2).

2. A fictitious forcing function $f_N(t)$ with arbitrary initial conditions $u_{No}, \dot{u}_{No}$, which the structure responds to, with $u_N(t), s_N(t)$.

Furthermore, if one considers the fictitious motion

$$u^*(t) = u_E(t) + u_N(t); \qquad s^*(t) = s_E(t) + s_N(t); \tag{12}$$

and a time history of the standard shear stress $s_o(t)$ such that

$$\bar{s}(t) = s^*(t) + s_o(t) \tag{13}$$

doesn’t violate eqn (10), that’s to say the yielding condition, on the duration of the motion T, *and that could be e.g.*

$$s_o(t) = \left[s_o' - s^*(t)\right]H\left[s^*(t) - s_o'\right] + \left[s_o'' - s^*(t)\right]H\left[s_o'' - s^*(t)\right] \tag{14}$$

one gets

$$\left[s(t) - s^*(t)\right]\left[\dot{u}(t) - \dot{u}_e(t) + \dot{u}^*(t) - \dot{u}^*(t)\right] - s_o(t)\dot{u}_p(t) \geq 0 \rightarrow \left[s(t) - s^*(t)\right]\left[\dot{u}(t) - \dot{u}^*(t)\right] - \left[s(t) - s^*(t)\right]\left[\dot{u}_e(t) - \dot{u}^*(t)\right] - s_o(t)\dot{u}_p(t) \geq 0 \tag{15}$$

and, by the virtual work principle, after some algebra,

$$f_N(\tau)\dot{u}(\tau)+s_o(\tau)\dot{u}_p(\tau)+\frac{1}{2}\varpi^2\frac{du_p^2(\tau)}{d\tau}\le f_N(\tau)\dot{u}^*(\tau)-\Delta\dot{T}^*(\tau)-\left(1+\frac{\varpi^2}{\omega_o^2}\right)\Delta\dot{L}^*(\tau)-\frac{\varpi^2}{\omega_o^2}\frac{d}{d\tau}\left\{\left[s(\tau)-s^*(\tau)\right]u_p(\tau)\right\} \tag{16}$$

whence one realises that the plastic displacement can be bounded ignoring the additional force $f_N(t)$ *and using the control component* $\varpi^2 u_p$ *as the fictitious force.*

Being $\Delta T^*(\tau)$, $\Delta L^*(\tau)$ respectively the (positively defined) kinetic and elastic energy of $\Delta u^*(\tau)=u(\tau)-u^*(\tau)$ and $\Delta\dot{L}^*(\tau)$ the time derivative of the latter, for $f_N(t)=0$ and the same initial conditions for the two motions, knowing that $u_p(0)=0$ and $s(T)=s^*(T)=0$, one gets

$$\frac{1}{2}\varpi^2u_p^2(t)\le-\int_0^t s_o(\tau)\dot{u}_p(\tau)d\tau \;\Rightarrow\; \int_0^t s_o(\tau)\dot{u}_p(\tau)d\tau\le 0\,. \tag{17}$$

Furthermore, with some additional algebra, applying the Schwartz inequality and normalising with respect to the energy of the forcing function, one can write

$$\frac{u_p^2(t)}{\|f(t)\|\sqrt{\int_0^t \dot{u}_p^2(\tau)d\tau}}\le\frac{2}{\varpi^2\|f(t)\|}\sqrt{\int_0^t s_o^2(\tau)d\tau}=F_{pl}(\mu,\varpi|f) \qquad \|f(t)\|=\sqrt{\int_0^t f^2(\tau)d\tau} \tag{18}$$

F_{pl} $(\mu,\varpi|f)$ has been shown [Baratta and Corbi, 1999a] to be a significant functional of the plastic response: actually, some numerical tests have been experienced on a s.d.o.f. structure with natural frequency ω_o=30 rad/sec, damping coefficient ζ=3% and limit shear stress values $s_0'=400$ cm/sec^2, $s_0''=-400$ cm/sec^2, subject to a sine wave function lasting 5 secs with pulsation ω_f=15 rad/sec and amplitude f_o,. Such experiment yields: i) a purely elastic motion for f_o = 200 cm/sec^2, ii) an initial plastic excursion followed by a steady-state elastic stabilization for f_o = 300 cm/sec^2, iii) a sequence of elastic perfectly plastic cycles for $f_o\ge$ 400 cm/sec^2; the results prove that F_{pl} $(\mu,\varpi|f)$ is an increasing function of the maximum plastic displacement u_{pmax} in the range of the damping control coefficient $0\le\mu\le1$ for different values of ϖ, whereas plastic excursions occur [Figure 2.a]; one can also notice that the slope of the maximum plastic displacement versus the functional is small when the plastic component of the response is marginal (i.e. f_0 =300 cm/sec^2), while it increases, when the plastic phenomenon is very significant (i.e. f_0 = 400, 500, 600 cm/sec^2), that's to say that the functional is pretty sensitive to some increase in the response plastic component. Moreover, as the instantaneous plastic rate is produced by directly absorbing energy from the ground acceleration

$$\int_0^t \dot{u}_p^2(\tau)d\tau\le\|f(t)\|^2\int_0^t\tau d\tau=\frac{1}{2}t^2\|f(t)\|^2 \tag{19}$$

one can get the final expression

$$u_p^2(t)\le\frac{\sqrt{2}}{\varpi^2}\sqrt{\int_0^t s_o^2(\tau)d\tau}\cdot t\cdot\|f(t)\|=N_{pl}(\mu,\varpi|f) \tag{20}$$

Hence, a possible control strategy could be to search for the minimum of the functional $N_{pl}(\mu,\varpi|f)$ while keeping the energy $C(\mu,\varpi|f)$ of the control force $c^*(t)$, normalised with respect to the excitation energy, under a desired threshold C_o and the optimum problem can, finally, be defined as follows

$$\begin{aligned}&\mathit{find} && N_{pl}(\mu,\varpi|f)=\mathit{min}\\ &\mathit{sub} && C(\mu,\varpi|f)\le C_o\end{aligned} \tag{21}$$

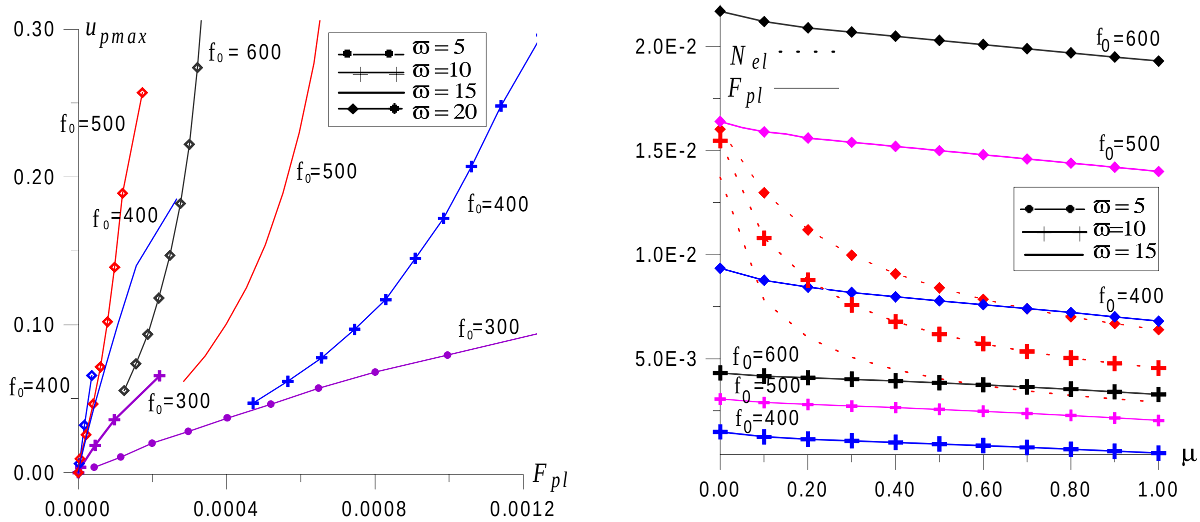

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Figure 2: a)The max plastic displac. u_{pmax} for $\mu \in [0,1]$ vs the plastic functional F_{pl} for various ϖ and f_o. b)The elastic N_{el} and plastic F_{pl} functionals versus μ for various ϖ and f_o.

EVALUATION OF THE OBJECTIVE FUNCTIONAL

If the fictitious forcing function is equal to zero, i.e. $f_N(t) = 0$, $s^*(t) = s_E(t)$, the functional can be expressed as

$$N_{pl}(\mu,\varpi|f) = \frac{\sqrt{2}}{\varpi^2}\sqrt{\int_0^t s_o^2(\tau)d\tau}\cdot t\cdot\|f(t)\| = \frac{\sqrt{2}}{\varpi^2}\sqrt{\left[\int_0^t s_o^2(\tau)d\tau\right]^+ + \left[\int_0^t s_o^2(\tau)d\tau\right]^-}\cdot t\cdot\|f(t)\| \tag{22}$$

where

$$\begin{aligned}\left[\int_0^t s_o^2(\tau)d\tau\right]^+ &= \int_0^t \left[Prob\{\tilde{s}_E(\tau) > s_o',\tau\}\right]\left[\tilde{s}_E(\tau) - s_o'\right]^2 d\tau = \int_0^t P^+(\tau)\left[\tilde{s}_E(\tau) - s_o'\right]^2 d\tau \le \int_{t^{*+}}^t P^+(\tau)\left[\left|\tilde{s}^+_{E\,max}(\tau)\right| - \left|s_o'\right|\right]^2 d\tau \\ \left[\int_0^t s_o^2(\tau)d\tau\right]^- &= \int_0^t \left[Prob\{\tilde{s}_E(\tau) < s_o'',\tau\}\right]\left[s_o'' - \tilde{s}_E(\tau)\right]^2 d\tau = \int_0^t P^-(\tau)\left[s_o'' - \tilde{s}_E(\tau)\right]^2 d\tau \le \int_{t^{*-}}^t P^-(\tau)\left[\left|\tilde{s}^-_{E\,max}(\tau)\right| - \left|s_o''\right|\right]^2 d\tau\end{aligned} \tag{23}$$

with t^{*+} and t^{*-} the instants at which the plastic threshold is exceeded in the two directions. As

$$\left|\tilde{s}^+_{E\,max}(\tau)\right| - \left|s_o'\right| \le \omega_o^2 \left\|h_c(\tau)\right\|\left\|f(\tau)\right\|(1-\gamma^+);\qquad \left|\tilde{s}^-_{E\,max}(\tau)\right| - \left|s_o''\right| \le \omega_o^2 \left\|h_c(\tau)\right\|\left\|f(\tau)\right\|(1-\gamma^-) \tag{24}$$

with γ^+, γ^- two design coefficients suitably defined, one gets

$$N_{pl}(\mu,\varpi|f) = t\sqrt{2}\left(\frac{\omega_o}{\varpi}\right)^2 \left\|h_c(t)\right\|\left\|f(t)\right\|^2 \sqrt{\left[(1-\gamma^+)\int_{t^{*+}}^t P^+(\tau)d\tau + (1-\gamma^-)\int_{t^{*-}}^t P^-(\tau)d\tau\right]} \tag{25}$$

Marking by $p_u(u_E, t)$ the marginal density of the displacement stochastic variable $\tilde{u}_E(t)$, and u_o', u_o'' the displacement values at the yielding threshold, one can express the probabilities $P^+(t)$ and $P^-(t)$ quoted in eqn (25)

$$Prob\{\tilde{u}_E(t) > u'_o, t\} = \int_{u'_o}^{+\infty} p_u(u_E, t)\, du_E = Prob\{\tilde{s}_E(t) > s'_o, t\} = P^+(t)\,;\; Prob\{\tilde{u}_E(t) < u''_o, t\} = \int_{-\infty}^{u''_o} p_u(u_E, t)\, du_E = Prob\{\tilde{s}_E(t) < s''_o, t\} = P^-(t)$$

(26)

with $p_u(u_E, t)$ having a suitable expression

$$p_u(u_E, t) = \begin{cases} \frac{1}{u_h(t)} \rho e^{-\frac{1}{1-\left[\frac{u_E(t)}{u_h(t)}\right]}} & if \quad u_E(t) < u_h(t) \\ 0 & if \quad u_E(t) > u_h(t) \end{cases} \qquad (27)$$

NUMERICAL RESULTS

The structural s.d.o.f. elasto-plastic model of Figure 1, with frequency $\omega_o = 30 \text{ rad}\cdot\text{sec}^{-1}$ and damping coefficient ζ=5%, and $u'_o = 2$ cm and $u''_o = -2$ cm, subject to an harmonic forcing function resulting by the combination of ten sine waves lasting 10 secs and acting at ω_f =15,18,21,24,27,30,33,36,39,42 rad/sec[1] is considered.

One refers for the uncontrolled response under the given excitation, and its comparison with the plastic controlled results to Table 2, which summarises some previously fulfilled numerical investigations [Baratta and Corbi, 1999b].

In Figure 4 the comparison of the total and plastic displacements one gets by following the above reported procedure with those deriving from the adoption of a norm strategy [Baratta and Voiello, 1985] and an “exact” plastic strategy is proposed. In details, in the latter case, one has assumed the objective function of the optimum numerical procedure coinciding with the maximum plastic displacement that one gets by operating the time simulation (in that the procedure is “exact”) on the linearly controlled model; the constraint condition consists, in this case, of bounding the maximum read value of the control force under the maximum it reaches by adopting the proposed approximated plastic control method; the problem, is hence

$$\begin{array}{ll} find & u_{pmax}(\mu, \varpi | f) = min \\ sub & c_{max}(\mu, \varpi | f) \le c_{omax} \end{array}$$

(28)

where c_{omax} is given in Table 2. It is clear that, whereas the forcing function induces in the structural system a main plastic component, the best performance in terms of response reduction is accomplished by the plastic strategy.

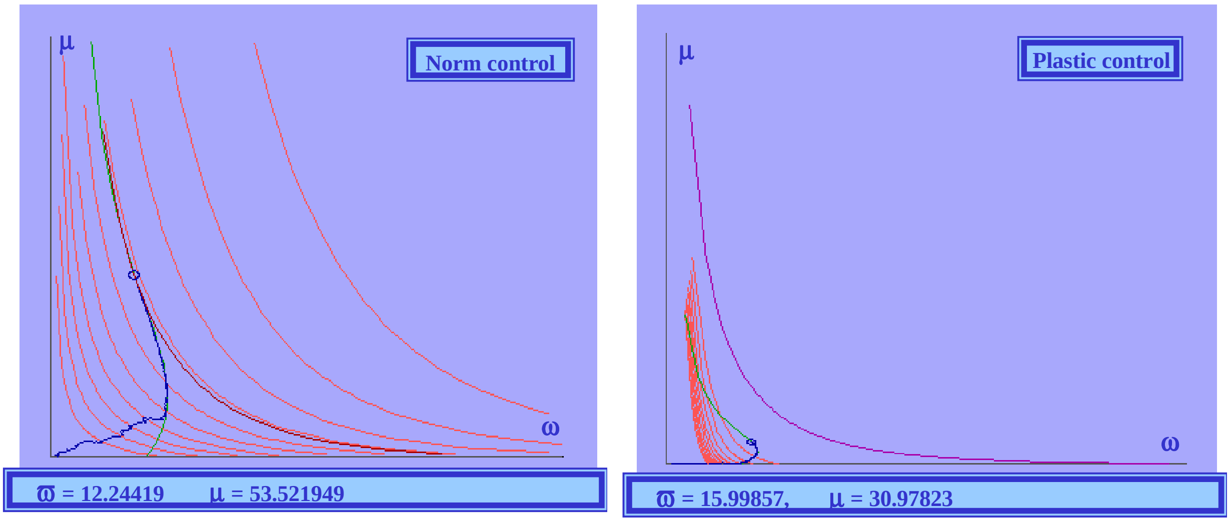
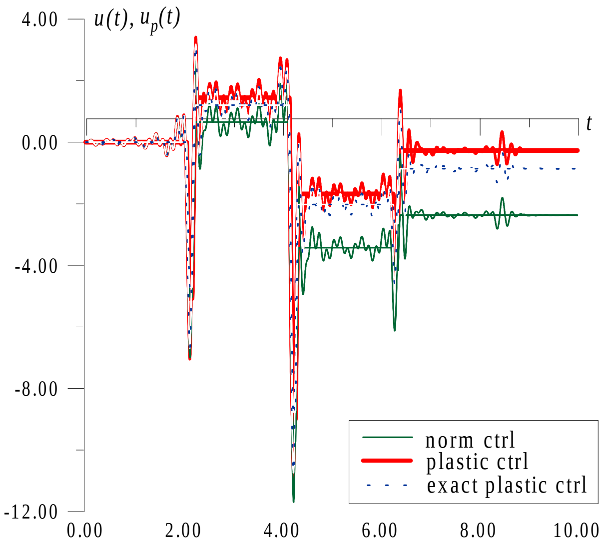


Figure 3: Constrained optimisation for the considered control approaches.

Table 2: Performance indexes for the considered control approaches.

	uncontrolled	plastic ctrl	u_{pmax} exact plastic	norm ctrl
u_{max}	16.35755	10.9734	10.7339	11.6911
u_{pmax}	14.3906	9.0072	8.7615	9.7182
c_{max}	0	**3626=** c_{omax}	3625	3232
$u(T)$	-4.7030	-0.2818	-0.8716	-2.3756
$u_p(T)$	-4.6721	-0.2758	-0.8654	-2.3695
$c(T)$	0	-76.7853	-204.6090	352.9349



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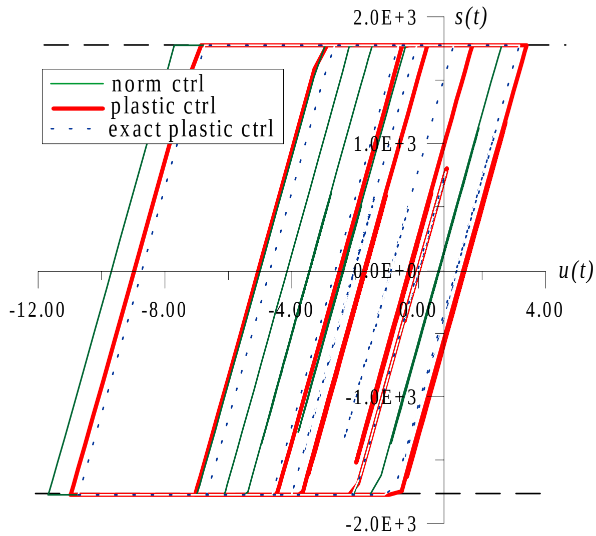

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Figure 4: Controlled response: a) total and plastic displacement; b) hysteretic cycles.

Actually the norm strategy, (an interesting comparison is proposed in Figure 2.b between $\boldsymbol{F_{pl}}$ and the elastic norm $\boldsymbol{N_{el}}$, assumed as the objective function in a supposedly purely elastic behaviour model [Baratta and Voiello, 1985]), by nearby using the same control force (Figure 5b), seems to fail when deepening into the plastic domain (the steady-state portion). On the other hand, the “exact” procedure achieves a lightly lower maximum displacement, while showing a worse performance, in the average, on the time vector length, when applying a control force pretty close to the plastic control one (Figure 5a). From Table 2 one can still emphasise that, for the “exact” control procedure a small reduction in terms of the maximum values of the response variables results in higher residual values at the end of the motion.

CONCLUSIONS

An optimality criterion for a linear control strategy in the elastic-plastic range has been proposed for an elastic-perfectly plastic s.d.o.f. structural model subject to a dynamic forcing function. The approach, finalised to reduce and mitigate the excursions in the plastic domain, is based on minimising the response exceeding the plastic threshold, while keeping bounded the energy of the control force. The adoption of the algorithm results in a deep reduction of the structure ductility demand and, if compared with control strategies assumed for the linear model and usually applied to non-linear behaviour models, in a much higher time and residual response mitigation.

An “exact” control strategy has also been considered, in order to evaluate a possible improvement produced by the exact solution of the differential equations system when minimising the real maximum plastic displacement: even in this case, due to the simply local action of this approach, the plastic algorithm has shown a visibly better performance and, definitively, an higher reliability with reference to structural systems with a main plastic component response.

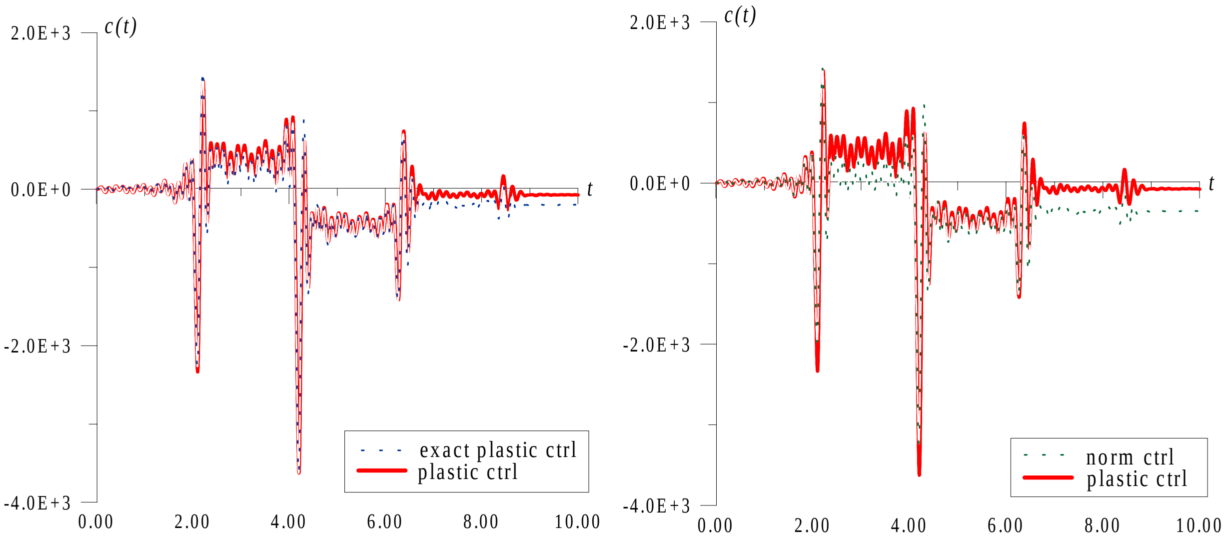

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Figure 5: The control force involved in the considered control approaches.

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