SEISMIC RESPONSE EVALUATION USING IMPULSE SERIES METHOD

ADRIAN I DANILIU[1] **And MIRELA G BATALAN**[2]

SUMMARY

Based-on mathematical considerations the impulse series method expresses the seismic response by analytical formulas, instead of integrals. It is mathematically proved that the input and output of a linear system can be reduced to specific random pulse trains. Six earthquake records are analysed, new parameters being calculated. Formulas for the mean and variance of the linear response of SDOF and MDOF systems are obtained. The capacity to assess qualitatively the deterministic and random response is an important feature of the impulse series method, a procedure recently developed in Romania.

INTRODUCTION

The response of linear systems to ground motions is usually convenient to be determined using Duhamel integral. Actual seismic acceleration is not an elementary function hence the seismic response can be only expressed in the form of an integral. In order to solve this type of integrals a step-by-step technique has to be considered. Numerical integration provides quantitative data needed to estimate seismic demands (displacement, energy, ductibility, force) for conceptual design. However there is a great need to express the seismic demands in terms of analytic formulas.

LINEAR DETERMINISTIC RESPONSE

The responses of SDOF and MDOF linear systems when subjected to earthquake ground motions are given by the following differential equations:

$$\ddot{x} + 2\xi\omega_0\dot{x} + \omega_0^2 x = -\ddot{u} \tag{1}$$

$$[M]\{\ddot{x}\} + [c]\{\dot{x}\} + [k]\{x\} = -[M]\{1\}\ddot{u} \tag{2}$$

where $x, \dot{x}, \ddot{x}$ are the relative displacement, velocity and acceleration of the SDOF system, $\ddot{u}$ =base acceleration, $\omega_0 =$ natural frequency, $\xi =$ viscous damping factor. $[M], [c], [k]$ are respectively the $(N \times N)$ mass, damping and stiffness matrices. The vectors $\{x\}, \{\dot{x}\}, \{\ddot{x}\}$ represent the displacements, velocities and accelerations of the MDOF systems, which has N degrees of freedom. If $x(0) = \dot{x}(0) = 0$ then the solution of Eq.(1) is Duhamel integral

$$x(t) = \int_0^t \ddot{u}(\tau)h(t-\tau)d\tau \tag{3}$$

where the impulse response function (IRF) of system is

[1] *ROMANIAN ASSOCIATION FOR EARTHQUAKE ENGINEERING (ARIS) Aleea Crizantemelor 6, Ap.3, 6800 BOTOSANI, ROMANIA*

[2] *Email: daniel_batalan@mail.petar.ro*

$$h(t-\tau)=\frac{1}{\omega_a}\theta(t-\tau)\exp[-\xi\omega_0(t-\tau)]\sin\omega_a(t-\tau);\ \theta(t-\tau)=\text{Heaviside function,}$$

$\omega_a=\omega_0\sqrt{1-\xi^2}$, t is a fixed instant of time and τ is the variable of integration. Dividing the interval $[0,t]$ into n subintervals $[t_{i-1},t_i]$ with t_i chosen so that $\ddot{u}(t_i)=0$ for $i=\overline{1,n-1}$, $t_n=t$, $\alpha_n=\alpha_n(t)$ and applying the first mean value theorem one obtains:

$$x(t)=\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\ddot{u}(\tau)h(t-\tau)d\tau=\sum_{i=1}^{n}U_i h(t-\alpha_i) \qquad (4)$$

in which $U_i=\int_{t_{i-1}}^{t_i}\ddot{u}(\tau)d\tau$, $U_n=U_n(t)$, $n=n(t)$ and $\alpha_i\in[t_{i-1},t_i)$ are not unique determined.

Turning to Eq.(2) using the modal transformation $\{x(t)\}=[v]\{q\}$ with $[v]=$ modal matrix, $\{q\}=$ vector of normal coordinates, the total displacement corresponding to the *l*-th mass is represented in the form

$$x_l(t)=\sum_{k=1}^{N}v_{lk}q_k=\sum_{k=1}^{N}v_{lk}\int_0^t\ddot{u}(\tau)b_k h_k(t-\tau)d\tau=\sum_{k=1}^{N}\sum_{i=1}^{n}U_i v_{lk}b_k h_k\left(t-\alpha_i^{(k)}\right) \qquad (5)$$

where $b_k=\sum_{s=1}^{N}v_{sk}m_s$ and $h_k(t)=\frac{1}{\omega_{ak}}\exp(-\xi_k\omega_k t)\sin\omega_{ak}t$; $m_s=$ mass at level s, ω_{ak} and ξ_k are the damped angular frequency and damping in the *k*-th mode. The existence of time instants $\alpha_i^{(k)}$ is proved by the first mean value theorem, $\alpha_i^{(k)}\in[t_{i-1},t_i)$. Table 1 lists the results of the analysis of six earthquake records via the impulse series method. The following notations are used: $U_{\max}^{+}=$ maximum positive impulse magnitude, $U_{\min}^{-}=$ minimum negative impulse magnitude, $\sum U_i^{+}$ and $\sum U_i^{-}$ are the sums of the positive and negative impulse magnitudes, $\sum U_i^2=$ the sum of the squares of the impulse magnitudes at the end of the earthquake.

Table 1

Earthquake record	Duration (sec)	$U_{\max}^{+}$ (cm/sec)	$U_{\min}^{-}$ (cm/sec)	$\sum U_i^{+}$ (cm/sec)	$\sum U_i^{-}$ (cm/sec)	$\sum U_i^2$ $(cm^2/\sec^2)$
Bucharest N-S, 1977	40	101.28	-96.20	357.59	-355.63	25011
El Centro N-S, 1940	29	56.57	-45.73	651.17	-657.83	22716
Bucharest 1986	22	12.06	-10.52	113.34	-116.35	1036
New Mexico N-S,1985	59	67.52	-74.76	894.61	-901.05	76849
Bucharest E-W, 1977	15	49.50	-30.50	224.65	-224.55	7472
Bucharest, vert., 1977	40	5.68	-11.64	141.36	-140.69	841

SEISMIC RESPONSE AS A GENERALIZED FUNCTION

The concept of generalized function (or distribution) permits the extension of the rules connected with differentiation and integration formulas without usual restrictions imposed on the behaviour of ordinary functions. In the field of distributions one may be proved (Daniliu 1996) that the continuous base acceleration may be substituted by a series of Dirac impulses applied at the time instants α_i. For a specified impulse response function $h(t-\tau)$ one can write in two symbolic forms:

$$x(t) = < \ddot{u}(\tau), h_t(\tau) > = \int_{-\infty}^{+\infty} \ddot{u}(\tau) h(t-\tau) d\tau = \sum_{i=1}^{n} U_i h(t-\alpha_i) \tag{6}$$

$$x(t) = < \sum_{i=1}^{n} U_i \delta(\tau-\alpha_i), h_t(\tau) > = \sum_{i=1}^{n} \int_{-\infty}^{+\infty} U_i \delta(\tau-\alpha_i) h(t-\tau) d\tau = \sum_{i=1}^{n} U_i h(t-\alpha_i)$$

and it follows $\sum_{i=1}^{n(t)} U_i \delta(\tau-\alpha_i) = \ddot{u}(\tau)$ (7)

The Fourier transform of $\ddot{u}$ is given by $U(\omega,t) = \int_{-\infty}^{+\infty} \sum_{i=1}^{n} U_i \delta(\tau-\alpha_i) e^{-j\omega\tau} d\tau = \sum_{i=1}^{n} U_i e^{-j\omega\alpha_i}$

where $j = \sqrt{-1}$, $n = n(t)$, $\alpha_i = \alpha_i(t)$, $U_n = U_n(t)$ and $U(\omega,t)$ is an evolutionary function of both *t* and ω. The mean square of $U(\omega,t)$ is obtained as

$$E[U^2] = E[U(\omega,t)U^*(\omega,t)] = E\left[\sum_{i=1}^{n(t)} U_i^2 + \sum_{i=1}^{n} \sum_{s=1}^{n} U_i U_s \cos\omega(\alpha_i - \alpha_s)\right] \approx E\left(\sum_{i=1}^{n(t)} U_i^2\right); \tag{8}$$

where $i \neq s$, *E* indicates the ensemble average and asterisk denote the complex conjugate. Here $\sum_{i=1}^{n(t)} U_i^2$ is an increasing function of time and plays the role of an input energy. In Fig. 1 is illustrated $\sum_{k=1}^{n} U_i \delta(\tau-\alpha_i)$ the corresponding impulse diagram to the 1977 Bucharest earthquake NOOS record (40 seconds).

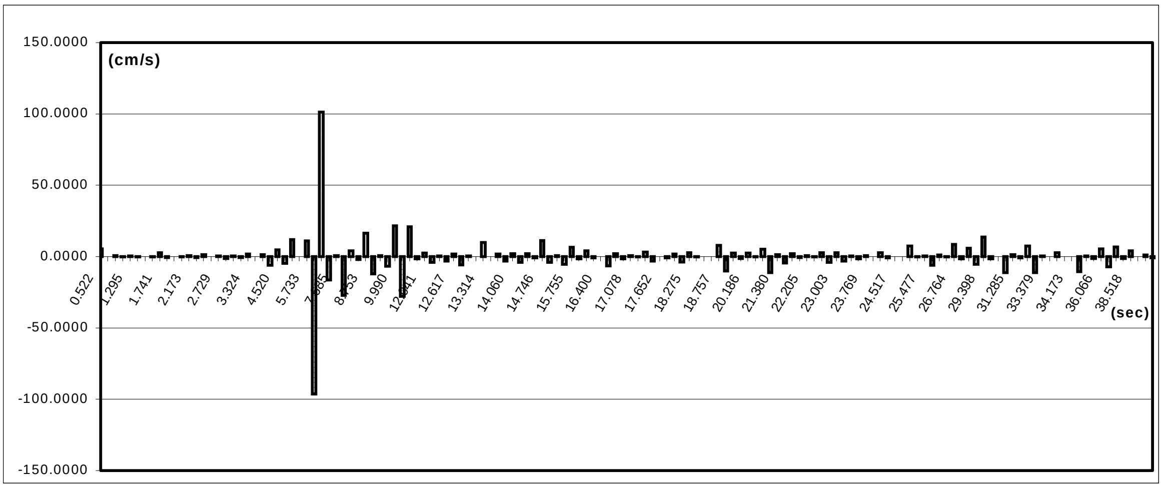

Fig. 1 Impulse diagram of the 1977 Bucharest earthquake.

RANDOM SEISMIC RESPONSE

Generally the responses of SDOF and MDOF systems are considered as random functions denoted by $X(t)$ and $X_l(t)$. The subintervals $[t_{i-1}, t_i]$ have random end-points and the differences $t_i - t_{i-1} = \varphi_i$ are considered as random independent variables. The number of impulses U_i is equal to the number of φ_i or the number of random variables α_i, with $i = \overline{1, n-1}$. This common number *n(t)* is a stochastic process, which can be treated as a stopped renewal process (Mihoc Gh. et al. 1978). The impulse magnitudes U_i are considered random variables with a deterministic envelope function called intensity function. In literature different intensity functions were proposed (for review see Barbat and Roca 1990, Lin and Cai 1997). The intensity function initiated by Amin and Ang has the expression:

$$I(t) = \begin{cases} 0 & , t < 0 \\ \left(\dfrac{t}{t_1}\right)^2 & , 0 \le t \le t_1 \\ 1 & , t_1 \le t \le t_2 \\ \exp[-\gamma(t - t_2)] & , t > t_2 \end{cases} \tag{9}$$

The intensity function proposed by Otto is given by:

$$I(t) = \frac{t}{t_0} e^{\left(1 - \frac{t}{t_0}\right)} \tag{10}$$

Returning to the Eq.(4) every pulse shape function may be approximatively evaluated as $h(t - \alpha_i) \approx p_1^i h(t - \tau_1^i) + p_2^i h(t - \tau_2^i) + \ldots + p_R^i h(t - \tau_R^i)$ where $\tau_1^i, \tau_2^i, \ldots, \tau_R^i$ are R_i points belonging to $[t_{i-1}, t_i]$, $p_r^i = \ddot{u}(\tau_r^i) \Big/ \sum_{r=1}^{R_i} \ddot{u}(\tau_r^i)$. The values $p_1^i, p_2^i, \ldots, p_R^i$ are bounded between zero and unity and their sum is $\sum_{r=1}^{R_i} p_r^i = 1$. The values p_r^i are playing the role of probability of α_i to be equal to τ_r^i, i.e. $p_r^i \approx P(\tau_r^i = \alpha_i)$. If $R_i \to \infty$ then it follows:

$$h(t - \alpha_i) = \int_{t_{i-1}}^{t_i} h(t - \tau) f_\alpha^{(i)}(\tau) d\tau \tag{11}$$

where $f_\alpha^{(i)}(\tau)$ may be compared with a probability density function of the random variable α_i, $p_r^i \approx f_\alpha^{(i)}(\tau)\Delta\tau$; $\int_{t_{i-1}}^{t_i} f_\alpha^{(i)}(\tau) d\tau = 1$. For the last subinterval $[t_{n-1}, t]$ some transformations can be done. Denoting t_{n+1} the next instant after t_{n-1} where $\ddot{u}(t_{n+1}) = 0$ and $\int_{t_{n-1}}^{t_{n+1}} \ddot{u}(\tau) d\tau = U_n^{\max}$ one may be shown that $\int_{t_{n-1}}^{t} \ddot{u}(\tau) h(t - \tau) d\tau = U_n(t) h(t - \alpha_n) = \gamma_n(t) U_n^{\max} h(t - \alpha_n) = U_n^{\max} h(t - \alpha_n^m)$ with $\alpha_n^m \in [t_{n-1}, t)$, $\gamma_n(t) \in (0,1]$. Hence $h(t - \alpha_n^m) \approx \sum_{r=1}^{R_n} p_r^n h(t - \tau_r^n)$ with $\tau_r^n \in [t_{n-1}, t)$ and $p_r^n = \ddot{u}(\tau_r^n) \Big/ \sum_{r=1}^{R_n^m} \ddot{u}(\tau_r^n)$,

R_n = the number of time instants in the subinterval $[t_{n-1}, t]$, R_n^m = the total number of instants in the subinterval $[t_{n-1}, t_{n+1}]$.

Instead of the Eq.(4) it might be written:

$$X(t) = \sum_{i=1}^{n(t)} \int_{t_{i-1}}^{t_i} U(\tau) z(\tau) f_\alpha^{(i)}(\tau) h(t-\tau) d\tau = \int_0^t U(\tau) z(\tau) f_\alpha(\tau) h(t-\tau) d\tau \qquad (12)$$

where $U(\tau) = |U_i|$, $z(\tau) = (-1)^{i-1}$, $f_\alpha^i(\tau) = f_\alpha(\tau)$ when $t_{i-1} \le \tau \le t_i, i = \overline{1,n}$. For the function $f_\alpha(\tau)$ one can adopt the hypothesis that $f_\alpha(\tau)\Delta\tau \approx n(\tau + \Delta\tau) - n(\tau)$. Passing to the limit it follows

$$f_\alpha(\tau) = \lim_{\Delta\tau \to 0} \frac{n(\tau + \Delta\tau) - n(\tau)}{\Delta\tau} = n'(\tau) \qquad (13)$$

when $n'(\tau)$ is the random arrival rate of the impulses. Calculating the square of $X(t)$ one obtains:

$$X^2(t) = \sum_{i=1}^{n} U_i^2 h^2(t - \alpha_i) + \sum_{i=1}^{n} \sum_{s=1}^{n} U_i U_s h(t-\alpha_i) h(t - \alpha_s), \qquad i \ne s \qquad (14)$$

.If *X(t)* is a zero mean process ($E[z(\tau)] = 0$ in Eq. 12) the variance of the seismic displacement response is calculated neglecting the cross correlation between $U_i h(t - \alpha_i)$ and $U_s h(t - \alpha_s)$.

$$E[X^2(t)] \approx E\left[\sum_{i=1}^{n} U_i^2 h^2(t - \alpha_i)\right] \qquad (15)$$

where the mean of the second term in the righ-hand side of Eq.(14) is considered zero. Making the same transformations as in Eqs.(11), (12) the variance becomes:

$$E[X^2(t)] = E\left[\sum_{i=1}^{n(t)} \int_{t_{i-1}}^{t_i} U^2(\tau) h^2(t-\tau) f_\alpha^{(i)}(\tau) d\tau\right] = E\left[\int_0^t U^2(\tau) h^2(t-\tau) f_\alpha(\tau) d\tau\right] =$$

$$= \int_0^t E[U^2(\tau)] h^2(t-\tau) E[f_\alpha(\tau)] d\tau = E[U_{max}^2] \int_0^t I^2(\tau) h^2(t-\tau) L(\tau) d\tau \qquad (16)$$

where $U_{max}^2 = \max|U_i|^2$, $\max|U_i|$ = maximum absolute impulse magnitude, *E*=mean operator, $I(\tau)$ is the intensity function, $L(\tau) = E[n'(\tau)]$. If the average impulse arrival rate $L(\tau)$ may be considered constant $L(\tau) = \lambda$ and denoting

$$\int_0^t h^2(t-\tau) I^2(\tau) d\tau = P_d(t) \qquad (17)$$

Eq.16 takes the form

$$E[X^2(t)] = P_d^2(t) \lambda E[U_{max}^2] \qquad (18)$$

The value of the function P_d depends on the choice of the intensity function. By substitution of Eq. (9) in Eq. (17) the maximum of P_d^2 is given by

$$\max\left[\mathrm{P}_d^2\right]=\int_0^{t_1}\left(\frac{\tau}{t_1}\right)^2 h^2(t-\tau)d\tau+\int_{t_1}^{t} h^2(t-\tau)d\tau=$$

$$=\frac{1}{2\omega_a^2}\left[e^{-2\xi\omega_0(t-\tau)}\left(\frac{1}{2\xi\omega_0}-\frac{2\xi\omega_0\cos 2\omega_a(t-\tau)-2\omega_a\sin 2\omega_a(t-\tau)}{(2\xi\omega_0)^2+(2\omega_a)^2}\right)\right]_{\tau=t}=$$

$$=\frac{1}{\xi\omega_0\left[(2\xi\omega_0)^2+(2\omega_a)^2\right]}\approx\frac{1}{4\xi\omega_0^3}=\frac{T_0^3}{32\pi^3\xi}\quad ;\ t_1\le t\le t_2 \qquad (19)$$

where $T_0=$natural period of the system. The coefficients P_d are computed by using Eq. (19) and are plotted in Fig.2 for $\xi=0.02; 0.05; 0.1$. If the intensity function (10) is adopted P_d^2 becomes:

$$\mathrm{P}_d^2=\int_0^t\frac{\tau^2}{t_0^2}e^{2\left(1-\frac{\tau}{t_0}\right)}h^2(t-\tau)d\tau \qquad (20)$$

After the numerical integration of this integral for $t_0=4$ sec, $\xi=0.05$, $T_0=0.5$ sec; 1 sec, the variation of P_d as a function of time is plotted in Fig.3.

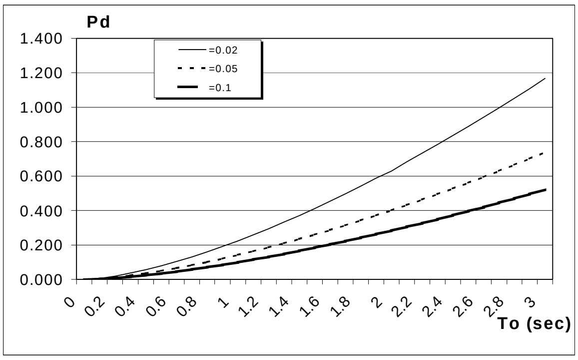

Fig.2

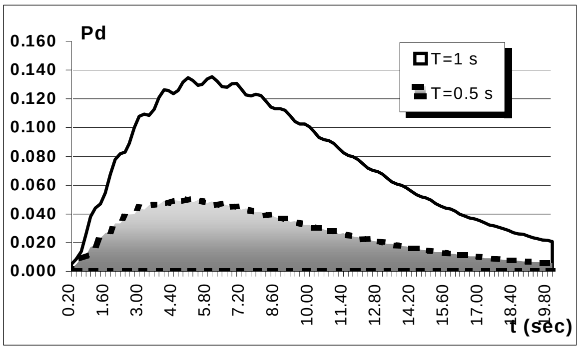

Fig.3

Concerning to MDOF systems similar relationships can be deduced. Starting from the Eq. (5) it may be written

$$h_k\left(t-\alpha_i^{(k)}\right)=\int_{t_{i-1}}^{t_i}h_k(t-\tau)f_{\alpha_i}^{(k)}(\tau)d\tau\ ;\quad h_k^2\left(t-\alpha_i^{(k)}\right)=\int_{t_{i-1}}^{t_i}h_k^2(t-\tau)f_{\alpha_i}^{(k)}(\tau)d\tau \qquad (21)$$

Calculating the square of $X_l(t)$ there results:

$$X_l^2(t)=\sum_{k=1}^{N}v_{lk}^2q_k^2+\sum_{k=1}^{N}\sum_{s=1}^{N}v_{lk}v_{ls}q_sq_k\ ; \qquad s\neq k \qquad (22)$$

In literature the contribution of the modal cross correlation is often neglected (Elishakoff 1983) then an approximation of the variance of $X_l(t)$ is given by:

$$E\left[X_l^2(t)\right] \approx \sum_{k=1}^{N} v_{lk}^2 E\left[q_k^2\right] = \sum_{k=1}^{N} v_{lk}^2 b_k E\left[\int_0^t U^2(\tau) h_k(t-\tau) f_\alpha^{(k)}(\tau) d\tau\right] =$$

$$= \sum_{k=1}^{n} v_{lk}^2 b_k^2 E\left[U_{\max}^2\right] \int_0^t I^2(\tau) h_k(t-\tau) L(\tau) d\tau \qquad (23)$$

where $E\left[f_\alpha^{(k)}(\tau)\right] = L(\tau)$. The Eqs. (16) and (23) were called Hertza formulas. Analogous to Eq. (18) the variance is

$$E\left[X_l^2(t)\right] = E\left[U_{\max}^2\right] \lambda \sum_{k=1}^{N} v_{lk}^2 b_k^2 \int_0^t I^2(\tau) h_k(t-\tau) d\tau = E\left[U_{\max}^2\right] \lambda \sum_{k=1}^{N} v_{lk}^2 b_k^2 \mathrm{P}_{dk}^2(t) \qquad (24)$$

where $L(\tau) = \lambda = const.$ and P_{dk} correspond to ω_k and ξ_k .

NONLINEAR ANALYSIS

Firstly a linear differential equation with variable coefficients will be considered

$$m\ddot{x} + c(t)\dot{x} + k(t)x = -m\ddot{u} \quad ; \quad x(0) = \dot{x}(0) = 0 \qquad (25)$$

where damping coefficient *c(t)* and stiffness *k(t)* are functions of time. The impulse response function depends on the time instant τ at which the Dirac impulse is applied:

$$mh''(t,\tau) + c(t)h'(t,\tau) + k(t)h(t,\tau) = \delta(t-\tau) \qquad (26)$$

Similarly to the equation with constant coefficients in this paper one proposes that for $h(t,\tau)$ should be chosen the expression:

$$h(t,\tau) = \frac{1}{m}\theta(t-\tau)A(t,\tau)\sin[\omega(t)(t-\tau)] \qquad (27)$$

where $A(t,\tau)$ is the variable amplitude, $\theta(t-\tau)=$Heaviside function and $\omega(t)$ is the variable pulsation of the system. Since the Eq. (25) is linear the superposition principle may be applied then, by the first mean value theorem, one obtains:

$$x(t) = \int_0^t \ddot{u}(\tau)A(t,\tau)\sin\omega(t)(t-\tau)d\tau = \sum_{i=1}^{n} U_i A(t,\alpha_i)\sin\omega(t)(t-\alpha_i) \qquad (28)$$

where $\alpha_i \in [t_{i-1}, t_i)$, $\omega^2(t) = k(t)/m$, $c(t)/m = 2\xi(t)\omega(t)$.

The initial condition for the derivative with respect to *t* of $h(t,\tau)$, $h'(\tau,\tau) = 1/m$ shows that $A(\tau,\tau) = 1/\omega(\tau)$. Physical considerations suggest that $A(t,\tau) < A(t-\tau,\omega(\tau))$ where the right term in the inequality is the amplitude of the IRF of the linear system with constant coefficients $\xi(\tau)$, $\omega(\tau)$. But the variation of the natural period and damping of actual structures depend on the time history of displacement *x(t)*. Hence a more realistic equation must be analysed:

$$m\ddot{x} + c(x,t)\dot{x} + k(x,t)x = -m\ddot{u} \qquad (29)$$

This equation is nonlinear and may be solved by quadratures only in the case of some particular forms of the coefficients.

For a stable system as a qualitative approximation one may try to denote *c(x,t)=C(t)* and *k(x,t)=K(t)* and the solution of (29) reduced to (28)

$$x(t) = \int_0^t \ddot{u}(\tau) h(t, \tau, x(\tau)) d\tau = \sum_{i=1}^{n} U_i h(t, \alpha_i, x(\alpha_i)) \tag{30}$$

which is an integral equation, the unknown function *x(t)* being included into the integral, $\alpha_i \in [t_{i-1}, t_i)$. The impulse response function depends on the whole time-history of *x(t)*. The Eq. (30) could be valid if additional conditions for the stability of equation are imposed. But these conditions are not yet elaborated in the general theory of differential equations. Now the losses of stiffness and plastic displacements of real structures have to be taken into account. The following equation of motion is proposed:

$$m\ddot{x} + c(x,t)\dot{x}_e + k(x,t)x_e = -m\ddot{u} \tag{31}$$

where *x(t)* the total displacement is divided into two parts $x_e(x,t) =$ elastic instantaneous displacement, $x_p(x,t) =$ plastic instantaneous displacement and $x(t) = x_e(x,t) + x_p(x,t)$.

The discussion on the evaluation of nonlinear response on the basis of Eqs. (28), (29), (31) is omitted for brevity but one can be found in Daniliu (1999).

CONCLUSIONS

Impulse series method simplifies the linear seismic analysis. For the first time the linear seismic response may be expressed in an analytical form replacing Duhamel integral. The estimation of random response via the impulse series method can lead to a more rational and transparent approach. In a well-known book (Lin and Cai 1997) it is recommended the modeling of nonstationary earthquake excitation by means of the random pulse train models. Based-on mathematical considerations, impulse series method rigorously establishes that the input and output of a linear system can be exactly reduced to a Dirac random impulse train and the output is equivalent to a random pulse train with trigonometric pulse shape functions..

REFERENCES

Barbat, A.H. and Roca, A. (1990). *Stochastic Prediction of Seismic Response Spectra,* Universitat Politecnica de Catalunya, Barcelona.

Cornea, I. et al. (1987). *Introducere in mecanica fenomenelor seismice si inginerie seismica,* Editura Academiei, Bucharest.

Daniliu, A.(1996). *Impulse Series Method in the Dynamic Analysis of Structures*. In: *Proceedings of the 11-th WCEE*, Acapulco, Mexico, Paper No. 577, Elsevier Science Ltd..

Daniliu, A. (1999). *Seismic Analysis of Structures by Impulse Series Method,* Ph. D. Thesis (in preparation), Faculty of Civil Engineering, Technical University of Iassy, Romania.

Elishakoff, I. (1983). *Probabilistic Methods in the Theory of Structures*. John Wiley and Sons, New York.

Lin, Y.K. and Cai, G.K. (1997) *Probabilistic Structural Dynamics,* Mc Graw-Hill, Inc., New-York.

Mihoc, Gh. et al. (1978) *Procese Stochastice*, Editura Stiintifica si Enciclopedica, Bucharest.