Technical Report: SBL-based Approaches for Quasi-Static and Doubly-Selective Sparse Channel Estimation for SISO/ MIMO-FBMC Systems

Prem Singh, Suraj Srivastava, Amrita Mishra, Aditya K. Jagannatham and Lajos Hanzo

I. Algorithmic Description: OMP

The algorithmic description of the OMP scheme is presented in Algorithm-1. The main steps of this algorithm are explained here. In each iteration i, Step-3 obtains the index j of the column of the matrix $\mathbf{D}$, which has the highest correlation with the residual vector $\mathbf{r}_{i-1}$. Step-4 updates the index set $\mathcal{I}_i$ by including the index j. Step-5 updates the matrix $\mathbf{D}^{\mathcal{I}}$ with the jth column of the matrix $\mathbf{D}$. The intermediate LS solution $\mathbf{h}^i$ is obtained in Step-6 and the corresponding residual vector $\mathbf{r}_i$ is computed in Step-7. The above procedure terminates, when the difference between the l_2-norm of the consecutive residues falls below a threshold ϵ_0.

Algorithm 1: OMP-based sparse channel estimation for SISO-FBMC systems

Input: Dictionary matrix $\mathbf{D}$, Observation vector $\mathbf{y}_0^{\text{td}}$, Stopping parameter ϵ_0

Output: Estimate $\widehat{\mathbf{h}}^{\text{OMP}}$ of the SISO-FBMC channel tap vector $\mathbf{h}$

1 **Initialization:** $\mathcal{I}_0 = [\ ]$, residue $\mathbf{r}_{-1} = \mathbf{0}$, $\mathbf{r}_0 = \mathbf{y}_0^{\text{td}}$, $\widehat{\mathbf{h}}^{\text{OMP}} = \mathbf{0}, \mathbf{D}^{\mathcal{I}} = [\ ]$, $i = 1$

2 **while** $(\| \mathbf{r}_{i-1} \|_2^2 - \| \mathbf{r}_{i-2} \|_2^2 \geq \ \epsilon_0)$ **do**

3 $\quad j = \underset{k=1,2,\ldots,L_h}{\operatorname{argmax}} \left| \mathbf{D}^H(:,k)\mathbf{r}_{i-1} \right|$

4 $\quad \mathcal{I}_i = \mathcal{I}_{i-1} \cup j$

5 $\quad \mathbf{D}^{\mathcal{I}} = \left[\mathbf{D}^{\mathcal{I}} \ \ \mathbf{D}(:,j) \right]$

6 $\quad \mathbf{h}^i = \left(\mathbf{D}^{\mathcal{I}} \right)^{\dagger} \mathbf{y}_0^{\text{td}}$

7 $\quad \mathbf{r}_i = \mathbf{y}_0^{\text{td}} - \mathbf{D}^{\mathcal{I}} \mathbf{h}^i$

8 $\quad i = i + 1$

9 **return**:$\widehat{\mathbf{h}}^{\text{OMP}} (\mathcal{I}_i) = \mathbf{h}^i$

II. Algorithmic Description: TD-SBL, TD-SBL-KF and TD-GSBL-KF

The algorithmic description for the proposed TD-SBL, TD-SBL-KF and TD-GSBL-KF schemes are presented below. Algorithm-2 describes the various steps in the TD-SBL scheme for estimation of the sparse SISO-FBMC CIR vector $\mathbf{h}$ using the procedure derived in Section-III-3

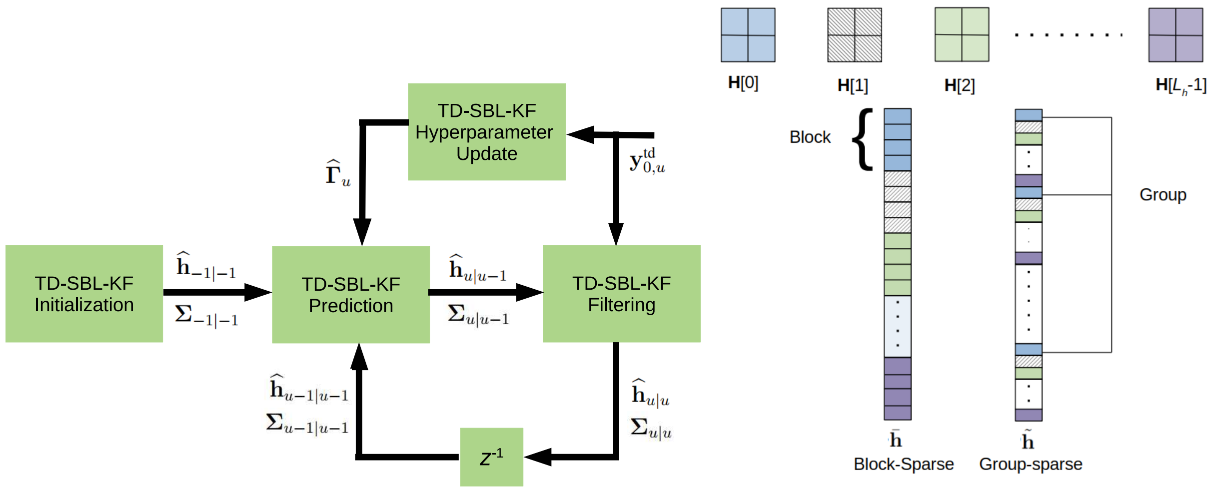

Fig. 1: (a) Block diagram representation of the TD-SBL-KF algorithm; (b) Block-sparse and group-sparse structures in the IAM and TD models, respectively.

of the manuscript. The various steps in the estimation of sparse doubly-selective SISO-FBMC CIR vector $\mathbf{h}_u$, using the TD-SBL-KF scheme developed in Section-IV, are similarly given in Algorithm-3. Its block diagram is also provided in Fig. 1(a). Finally, the proposed TD-GSBL-KF technique is described in Algorithm-4. Furthermore, for reader's convenience, the block-sparse and group-sparse structures arising in the IAM and TD models, respectively, have been shown in Fig. 1(b).

Note that the proposed SBL-KF framework initializes the hyperparameter matrix $\widehat{\mathbf{\Gamma}}_u^{(0)}$ for block u as $\widehat{\mathbf{\Gamma}}_u^{(0)} = \widehat{\mathbf{\Gamma}}_{u-1}^{(p)}$, i.e., to the converged estimate of the hyperparameter matrix obtained from the previous block. The advantage of this initialization procedure is two-fold: when the sparsity profile of the CIR does not change, the convergence is faster. On the other hand, when it changes suddenly, the proposed SBL-KF is able to detect the change and it adapts to the new sparsity profile in a few iterations. This has been illustrated via a simulation result in Fig. 2 of this document. From this figure, it can be readily observed that when the sparsity profile changes at the block index $u = 10$, the proposed SBL-KF-based schemes are able to adapt to the new profile within 5 blocks.

III. Computational Complexity

A comprehensive analysis is presented for the computational complexities of the proposed OMP, SBL and SBL-KF schemes for sparse channel estimation in SISO- and MIMO-FBMC systems. The complexity of each scheme is quantified in terms of complex additions and multiplications.

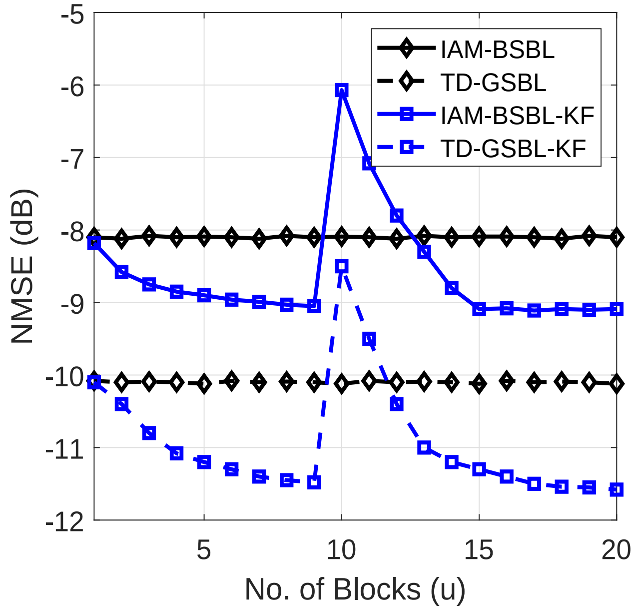

Fig. 2: NMSE versus number of blocks of the proposed schemes when sparsity profile changes at $u = 10$ with $N = 64$, $N_p = 28$ and $z = 3$, $\rho = 0.9981$ and SNR = 15 dB.

A. *Complexity of the OMP Scheme*

For a general channel estimation model, $\mathbf{y} = \mathbf{\Phi}\mathbf{h} + \boldsymbol{\eta}$ with $\mathbf{\Phi} \in \mathbb{C}^{M \times N}$, the computational complexity of the OMP scheme is derived next. Table-I details the computational cost of the various steps in the ith iteration, as described in Algorithm-1 of this document.

B. *Complexity of SBL Scheme for Quasi-Static Channel Estimation*

For a general channel estimation model, $\mathbf{y} = \mathbf{\Phi}\mathbf{h} + \boldsymbol{\eta}$ with $\mathbf{\Phi} \in \mathbb{C}^{M \times N}$, the computational complexity is derived next. The E-step in the SBL scheme requires the computation of mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{\Sigma}$ given below

$$\boldsymbol{\mu} = \mathbf{\Sigma}\mathbf{\Phi}^H\mathbf{R}_\eta^{-1}\mathbf{y} \text{ and } \mathbf{\Sigma} = \left(\mathbf{\Gamma}^{-1} + \mathbf{\Phi}^H\mathbf{R}_\eta^{-1}\mathbf{\Phi}\right)^{-1}, \tag{1}$$

where $\mathbf{R}_\eta$ denotes noise covariance matrix and the diagonal matrix $\mathbf{\Gamma}$ comprises of hyperparameters γ_i. In the M-step, the hyperparameter γ_i is updated as follows

$$\gamma_i = \left|\boldsymbol{\mu}(i)\right|^2 + \mathbf{\Sigma}(i,i). \tag{2}$$

Table-II details the computational cost of the various steps.

Algorithm 2: TD-SBL based sparse channel estimation in SISO-FBMC systems

Input: Observation $\mathbf{y}_0^{\text{td}}$, Dictionary Matrix $\mathbf{D}$, Noise Covariance $\mathbf{R}_\eta$, Stopping Parameters ϵ_1 and $N_{\max}$

Output: Estimate $\widehat{\mathbf{h}}^{\text{TD-SBL}}$ of the SISO-FBMC CIR vector $\mathbf{h}$

Initialization: $\widehat{\gamma}_i^{(0)} = 1, \forall\ 1 \leq i \leq L_h - 1 \implies \widehat{\boldsymbol{\Gamma}}^{(0)} = \mathbf{I}_{L_h}$, Set counter $p = 0$ and $\widehat{\boldsymbol{\Gamma}}^{(-1)} = \mathbf{0}_{L_h \times L_h}$

while $\left(\left\|\widehat{\boldsymbol{\Gamma}}^{(p)} - \widehat{\boldsymbol{\Gamma}}^{(p-1)}\right\|_F > \epsilon_1 \ \&\& \ p < N_{\max}\right)$ **do**

- $p \leftarrow p + 1$
- **E-step:** Evaluate *a posteriori* mean and covariance
  $$\boldsymbol{\Sigma}^{(p)} = \left(\left(\widehat{\boldsymbol{\Gamma}}^{(p-1)}\right)^{-1} + \mathbf{D}^H \mathbf{R}_\eta^{-1} \mathbf{D}\right)^{-1}$$
- $\boldsymbol{\mu}^{(p)} = \boldsymbol{\Sigma}^{(p)} \mathbf{D}^H \mathbf{R}_\eta^{-1} \mathbf{y}_0^{\text{td}}$
- **M-step:** Evaluate estimate of the hyperparameters
- **for** $i = 0, 1, \ldots, L_h - 1$ **do**
  - $\widehat{\gamma}_i^{(p)} = \left|\boldsymbol{\mu}^{(p)}(i)\right|^2 + \boldsymbol{\Sigma}^{(p)}(i, i)$
- $\widehat{\boldsymbol{\Gamma}}^{(p)} = \operatorname{diag}\left(\widehat{\gamma}_0^{(p)}, \widehat{\gamma}_2^{(p)}, \ldots, \widehat{\gamma}_{L_h-1}^{(p)}\right)$

return:$\widehat{\mathbf{h}}^{\text{TD-SBL}} = \boldsymbol{\mu}^{(p)}$

C. Complexity of SBL-KF Scheme for Doubly-Selective Channel Estimation

For a general channel estimation model, $\mathbf{y}_u = \boldsymbol{\Phi}\mathbf{h}_u + \boldsymbol{\eta}_u$ with $\boldsymbol{\Phi} \in \mathbb{C}^{M \times N}$ and state model $\mathbf{h}_u = \rho \mathbf{h}_{u-1} + \sqrt{1-\rho^2}\ \mathbf{w}_u$, where ρ denotes temporal correlation parameters and the vector $\mathbf{w}_u$ symbolises the innovation noise. For this model, the computational complexity is derived next. The E- and M-step of the SBL-KF scheme are identical to the SBL scheme, where as the former additionally requires following computations.

Let $\widehat{\mathbf{h}}_{u-1|u-1}$ and $\boldsymbol{\Sigma}_{u-1|u-1}$ represent the filtered estimate and the corresponding estimation error covariance matrix of the vector $\mathbf{h}_{u-1}$, respectively. Let $\widehat{\boldsymbol{\Gamma}}_u$ denote the estimate of the hyperparameter matrix $\boldsymbol{\Gamma}_u$ in the uth block. The MMSE prediction $\widehat{\mathbf{h}}_{u|u-1}$ of the vector $\mathbf{h}_u$ and the corresponding error covariance matrix $\boldsymbol{\Sigma}_{u|u-1}$ are obtained as

$$\widehat{\mathbf{h}}_{u|u-1} = \rho\ \widehat{\mathbf{h}}_{u-1|u-1}, \quad \text{and} \quad \boldsymbol{\Sigma}_{u|u-1} = \rho^2\ \boldsymbol{\Sigma}_{u-1|u-1} + (1-\rho^2)\ \widehat{\boldsymbol{\Gamma}}_u. \tag{3}$$

Furthermore, the filtered estimate $\widehat{\mathbf{h}}_{u|u}$ and the associated error covariance $\boldsymbol{\Sigma}_{u|u}$ can be updated as

$$\widehat{\mathbf{h}}_{u|u} = \widehat{\mathbf{h}}_{u|u-1} + \mathbf{K}_u(\mathbf{y}_u - \boldsymbol{\Phi}\widehat{\mathbf{h}}_{u|u-1}), \quad \text{and} \quad \boldsymbol{\Sigma}_{u|u} = (\mathbf{I} - \mathbf{K}_u\boldsymbol{\Phi})\boldsymbol{\Sigma}_{u|u-1}, \tag{4}$$

Algorithm 3: TD-SBL-KF based sparse doubly-selective channel estimation in SISO-FBMC systems

Input: Observation $\mathbf{y}_{0,u}^{\text{td}}$, dictionary matrix $\mathbf{D}$, correlation coefficient ρ, noise covariance $\mathbf{R}_\eta$, stopping parameters ϵ_1 and $N_{\max}$

Output: Estimate $\widehat{\mathbf{h}}^{\text{TD-SBL-KF}}$ of the doubly-selective SISO-FBMC CIR vector $\mathbf{h}_u$

Initialization: $\widehat{\boldsymbol{\Gamma}}_{-1}^{\text{conv}} = \mathbf{I}_{L_h}$

for $u = 0, 1, 2, \ldots$ **do**

  Set $p = 0$, $\widehat{\boldsymbol{\Gamma}}_u^{(0)} = \widehat{\boldsymbol{\Gamma}}_{u-1}^{\text{conv}}$ and $\widehat{\boldsymbol{\Gamma}}_u^{(-1)} = \mathbf{0}_{L_h \times L_h}$

  while $\left(\left\|\widehat{\boldsymbol{\Gamma}}_u^{(p)} - \widehat{\boldsymbol{\Gamma}}_u^{(p-1)}\right\|_F > \epsilon_1 \;\&\&\; p < N_{\max}\right)$ **do**

    $p \leftarrow p + 1$

    $\boldsymbol{\Sigma}_u^{(p)} = \left(\mathbf{D}^H \mathbf{R}_\eta^{-1} \mathbf{D} + \left(\widehat{\boldsymbol{\Gamma}}_u^{(p-1)}\right)^{-1}\right)^{-1}$

    $\boldsymbol{\mu}_u^{(p)} = \boldsymbol{\Sigma}_u^{(p)} \mathbf{D}^H \mathbf{R}_\eta^{-1} \mathbf{y}_{0,u}^{\text{td}}$

    for $i = 0, 1, \ldots, L_h - 1$ **do**

      $\widehat{\gamma}_{i,u}^{(p)} = \left|\boldsymbol{\mu}_u^{(p)}(i)\right|^2 + \boldsymbol{\Sigma}_u^{(p)}(i,i)$

    $\widehat{\boldsymbol{\Gamma}}_u^{\text{conv}} = \widehat{\boldsymbol{\Gamma}}_u^{(p)} = \text{diag}\left(\widehat{\gamma}_{0,u}^{(p)}, \widehat{\gamma}_{1,u}^{(p)}, \ldots, \widehat{\gamma}_{L_h-1,u}^{(p)}\right)$

  if $(u == 0)$ **then**

    Set $\widehat{\mathbf{h}}_{-1|-1} = \mathbf{0}_{L_h \times 1}, \boldsymbol{\Sigma}_{-1|-1} = \widehat{\boldsymbol{\Gamma}}_0^{(p)}$

  Obtain $\widehat{\mathbf{h}}_{u|u-1}$ and $\boldsymbol{\Sigma}_{u|u-1}$ using (27)

  Obtain the Kalman-gain $\mathbf{K}_u$ using (29)

  Obtain $\widehat{\mathbf{h}}_{u|u}$ and $\boldsymbol{\Sigma}_{u|u}$ using (28)

  return:$\widehat{\mathbf{h}}_u^{\text{TD-SBL-KF}} = \widehat{\mathbf{h}}_{u|u}$

where $\mathbf{K}_u$ is the Kalman-gain matrix given by $\mathbf{K}_u = \boldsymbol{\Sigma}_{u|u-1}\boldsymbol{\Phi}^H(\mathbf{R}_\eta + \boldsymbol{\Phi}\boldsymbol{\Sigma}_{u|u-1}\boldsymbol{\Phi}^H)^{-1}$. Table-III details the computational cost of the various steps for the SBL-KF scheme.

IV. Bayesian Cramér-Rao Lower Bounds and Complexity Analysis

The BCRLBs of the TD model of CSI estimation in MIMO-FBMC systems using (29) and (50) are derived next.

1) BCRLB of Quasi-static MIMO-FBMC channel estimation: The Bayesian Fisher information matrix (FIM) denoted by $\mathbf{J}_B \in \mathbb{C}^{N_r N_t L_h \times N_r N_t L_h}$ for the estimate of the CSI $\tilde{\mathbf{h}}$ can be

Algorithm 4: TD-GSBL-KF based sparse doubly-selective channel estimation in MIMO-FBMC systems

Input: $\bar{\mathbf{y}}_{0,u}^{\text{td}}$, $\bar{\boldsymbol{\mathcal{D}}}$, ρ, $\tilde{\mathbf{R}}_{\eta}$, ϵ_1 and $N_{\max}$

Output: Estimate $\widehat{\mathbf{h}}_{u}^{\text{TD-GSBL-KF}}$ of $\tilde{\mathbf{h}}_u$

Initialization: $\widehat{\boldsymbol{\Gamma}}_{-1}^{\text{conv}} = \mathbf{I}_{L_h}$

for $u = 0, 1, 2, \ldots$ **do**

  Set $p = 0$, $\widehat{\boldsymbol{\Gamma}}_{u}^{(0)} = \widehat{\boldsymbol{\Gamma}}_{u-1}^{\text{conv}}$ and $\widehat{\boldsymbol{\Gamma}}_{u}^{(-1)} = \mathbf{0}_{L_h \times L_h}$

  while $\left(\left\| \widehat{\boldsymbol{\Gamma}}_{u}^{(p)} - \widehat{\boldsymbol{\Gamma}}_{u}^{(p-1)} \right\|_F > \epsilon_1 \ \&\& \ p < N_{\max} \right)$ **do**

    $p \leftarrow p + 1$

    $\tilde{\boldsymbol{\Sigma}}^{(p)} = \left(\bar{\boldsymbol{\mathcal{D}}}^H \tilde{\mathbf{R}}_{\eta}^{-1} \bar{\boldsymbol{\mathcal{D}}} + \left(\mathbf{I}_{N_r N_t} \otimes \widehat{\boldsymbol{\Gamma}}_{u}^{(p-1)} \right)^{-1} \right)^{-1}$; $\tilde{\boldsymbol{\mu}}^{(p)} = \tilde{\boldsymbol{\Sigma}}^{(p)} \bar{\boldsymbol{\mathcal{D}}}^H \tilde{\mathbf{R}}_{\eta}^{-1} \bar{\mathbf{y}}_{0,u}^{\text{td}}$

    for $i = 0, 1, \ldots, L_h - 1$ **do**

      $\widehat{\gamma}_{i,u}^{(p)} = \frac{1}{N_r N_t} \sum_{d=1}^{N_r N_t} \left| \tilde{\boldsymbol{\mu}}^{(p)}(\tilde{d}) \right|_2^2 + \tilde{\boldsymbol{\Sigma}}^{(p)}(\tilde{d}, \tilde{d})$ where $\tilde{d} = (i + 1 + (d-1) L_h)$

    $\widehat{\boldsymbol{\Gamma}}_{u}^{\text{conv}} = \widehat{\boldsymbol{\Gamma}}_{u}^{(p)} = \operatorname{diag}\left(\widehat{\gamma}_{0,u}^{(p)}, \widehat{\gamma}_{1,u}^{(p)}, \ldots, \widehat{\gamma}_{L_h-1,u}^{(p)} \right)$

  if $(u == 0)$ **then**

    $\widehat{\mathbf{h}}_{-1|-1} = \mathbf{0}_{N_r N_t L_h \times 1}; \boldsymbol{\Sigma}_{-1|-1} = \mathbf{I}_{N_r N_t} \otimes \widehat{\boldsymbol{\Gamma}}_{0}^{(p)}$

  Obtain $\widehat{\mathbf{h}}_{u|u-1}$ and $\boldsymbol{\Sigma}_{u|u-1}$ using (3)

  Obtain $\widehat{\mathbf{h}}_{u|u}$ and $\boldsymbol{\Sigma}_{u|u}$ using (4)

  return:$\widehat{\mathbf{h}}_{u}^{\text{TD-GSBL-KF}} = \widehat{\mathbf{h}}_{u|u}$

TABLE I: Complexity of OMP scheme

Operation	Complex Multiplications	Complex Additions
Step-3	NM	$N(M-1)$
Step-6	$\frac{1}{2}i^3 + \frac{5}{2}i^2 + iM$	$\frac{1}{2}i^3 - i^2\left(M - \frac{1}{2}\right) + iM - 2i$
Step-7	Mi	Mi

expressed as [1]

$$\mathbf{J}_B = \mathbf{J}_D + \mathbf{J}_P, \tag{5}$$

TABLE II: Complexity of SBL scheme

Operation	Complex Multiplications	Complex Additions
$\boldsymbol{\Sigma}$	$\frac{M^3}{2}+\frac{N^3}{2}+\frac{3M^2}{2}+\frac{3N^2}{2}+NM^2+N^2M+N$	$\frac{M^3}{2}+\frac{N^3}{2}-\frac{M^2}{2}-\frac{3N^2}{2}+NM^2+N^2M+N-MN$
$\boldsymbol{\mu}$	N^2+NM	$N^2+NM-2N$
γ_i	N	N

TABLE III: Complexity of SBL-KF scheme

Operation	Complex Multiplications	Complex Additions
$\boldsymbol{\Sigma}$	$\frac{M^3}{2}+\frac{N^3}{2}+\frac{3M^2}{2}+\frac{3N^2}{2}+NM^2+N^2M+N$	$\frac{M^3}{2}+\frac{N^3}{2}-\frac{M^2}{2}-\frac{3N^2}{2}+NM^2+N^2M+N-MN$
$\boldsymbol{\mu}$	N^2+NM	$N^2+NM-2N$
γ_i	N	N
$\widehat{\mathbf{h}}_{u\vert u-1}$	N	-
$\boldsymbol{\Sigma}_{u\vert u-1}$	N^2+N	N
$\mathbf{K}_u$	$N^2M+2M^2N+\frac{M^3}{2}+\frac{3M^2}{2}$	$N^2M+2M^2N+\frac{M^3}{2}-\frac{M^2}{2}-2MN$
$\widehat{\mathbf{h}}_{u\vert u}$	$2MN$	$2MN$
$\boldsymbol{\Sigma}_{u\vert u}$	N^3+N^2M	$N^3+N^2M+N-2N^2$

where both $\mathbf{J}_D$ and $\mathbf{J}_P$ having the size $N_rN_tL_h \times N_rN_tL_h$, symbolize the FIMs associated with the pilot output $\bar{\mathbf{y}}_0^{\text{td}}$ and CSI $\tilde{\mathbf{h}}$, respectively. These matrices are formulated as

$$\mathbf{J}_D = -\mathbb{E}_{\tilde{\mathbf{h}},\bar{\mathbf{y}}_0^{\text{td}}}\left\{\frac{\partial^2 \log\, f\left(\bar{\mathbf{y}}_0^{\text{td}} \mid \tilde{\mathbf{h}}\right)}{\partial\tilde{\mathbf{h}}\partial\tilde{\mathbf{h}}^H}\right\} \quad \text{and} \quad \mathbf{J}_P = -\mathbb{E}_{\tilde{\mathbf{h}}}\left\{\frac{\partial^2 \log\, f\left(\tilde{\mathbf{h}};\boldsymbol{\Gamma}\right)}{\partial\tilde{\mathbf{h}}\partial\tilde{\mathbf{h}}^H}\right\}.$$

The quantities $\log\, p\left(\bar{\mathbf{y}}_0^{\text{td}} \mid \tilde{\mathbf{h}}\right)$ and $\log\, p\left(\tilde{\mathbf{h}};\boldsymbol{\Gamma}\right)$ are simplified to

$$\log\, f\left(\bar{\mathbf{y}}_0^{\text{td}} \mid \tilde{\mathbf{h}}\right) = \mathcal{C}_2 - \left(\bar{\mathbf{y}}_0^{\text{td}} - \bar{\boldsymbol{\mathcal{D}}}\tilde{\mathbf{h}}\right)^H \tilde{\mathbf{R}}_\eta^{-1}\left(\bar{\mathbf{y}}_0^{\text{td}} - \bar{\boldsymbol{\mathcal{D}}}\tilde{\mathbf{h}}\right), \text{ and}$$
$$\log\, f\left(\tilde{\mathbf{h}};\boldsymbol{\Gamma}\right) = \mathcal{C}_3 - \tilde{\mathbf{h}}^H(\mathbf{I}\otimes\boldsymbol{\Gamma})^{-1}\tilde{\mathbf{h}},$$

where the constants $\mathcal{C}_2$ and $\mathcal{C}_3$ are given by $\mathcal{C}_2 = -N_pN_r\log\pi - \log\,\det\left(\tilde{\mathbf{R}}_\eta\right)$ and $\mathcal{C}_3 = -N_tN_rL_h\log\pi - \log\,\det\left(\mathbf{I}_{N_rN_t}\otimes\boldsymbol{\Gamma}\right)$. Upon employing the above expressions, the FIMs $\mathbf{J}_D$ and $\mathbf{J}_P$ can be expressed as $\mathbf{J}_D = \bar{\boldsymbol{\mathcal{D}}}^H\tilde{\mathbf{R}}_\eta^{-1}\bar{\boldsymbol{\mathcal{D}}}$ and $\mathbf{J}_P = \left(\mathbf{I}_{N_rN_t}\otimes\boldsymbol{\Gamma}\right)^{-1}$. Substituting these into (5), the Bayesian FIM $\mathbf{J}_B$ is expressed as

$$\mathbf{J}_B = \bar{\boldsymbol{\mathcal{D}}}^H\tilde{\mathbf{R}}_\eta^{-1}\bar{\boldsymbol{\mathcal{D}}} + \left(\mathbf{I}_{N_rN_t}\otimes\boldsymbol{\Gamma}\right)^{-1}. \tag{6}$$

It can be readily observed from the above expression that the Bayesian FIM $\mathbf{J}_B$ follows $\mathbf{J}_{B,\text{SNR}_1} \succeq \mathbf{J}_{B,\text{SNR}_2}$, if $\text{SNR}_1 \geq \text{SNR}_2$, since the quantity $\bar{\boldsymbol{\mathcal{D}}}^H\tilde{\mathbf{R}}_\eta^{-1}\bar{\boldsymbol{\mathcal{D}}}$, in which $\bar{\mathbf{D}}$ comprises pilot symbols and $\mathbf{R}_\eta$ is the noise covariance matrix, is a function of SNR. This implies that $\mathbf{J}_{B,\text{SNR}_1}^{-1} \preceq \mathbf{J}_{B,\text{SNR}_2}^{-1}$, which in turn implies that $\mathrm{Tr}\left\{\mathbf{J}_{B,\mathrm{SNR}_1}^{-1}\right\} \leq \mathrm{Tr}\left\{\mathbf{J}_{B,\mathrm{SNR}_2}^{-1}\right\}$. This verifies the decreasing trend of mean square error (MSE) with SNR. The MSE of the estimate of the vectorized channel $\tilde{\mathbf{h}}$ can in turn be formulated as

$$\mathrm{MSE}\left(\tilde{\mathbf{h}}^{\text{TD-GSBL}}\right) \geq \mathrm{Tr}\left\{\mathbf{J}_B^{-1}\right\}. \tag{7}$$

2) BCRLB of Doubly-selective MIMO-FBMC channel estimation: Let $\mathbf{J}_u \in \mathbb{C}^{N_r N_t L_h \times N_r N_t L_h}$ denote the Bayesian FIM for the MSE of the doubly-selective CSI vector $\tilde{\mathbf{h}}_u$ estimate in the uth block. Using the results of [1], $\mathbf{J}_u$ can be recursively updated as

$$\mathbf{J}_u = \mathbf{G}_u^{22} - \mathbf{G}_u^{21}\left(\mathbf{J}_{u-1} + \mathbf{G}_u^{11}\right)^{-1}\mathbf{G}_u^{12}, \tag{8}$$

where the matrices $\mathbf{G}_u^{11}, \mathbf{G}_u^{12}, \mathbf{G}_u^{21}$ and $\mathbf{G}_u^{22}$, each of size $N_r N_t L_h \times N_r N_t L_h$, are formulated as

$$\mathbf{G}_u^{11} = -\mathbb{E}\left\{\frac{\partial^2 \mathcal{L}(\tilde{\mathbf{h}}_u \mid \tilde{\mathbf{h}}_{u-1})}{\partial \tilde{\mathbf{h}}_{u-1} \partial \tilde{\mathbf{h}}_{u-1}^H}\right\} \quad \text{and} \quad \mathbf{G}_u^{12} = -\mathbb{E}\left\{\frac{\partial^2 \mathcal{L}(\tilde{\mathbf{h}}_u \mid \tilde{\mathbf{h}}_{u-1})}{\partial \tilde{\mathbf{h}}_{u-1} \partial \tilde{\mathbf{h}}_u^H}\right\} = \left(\mathbf{G}_u^{21}\right)^H, \tag{9}$$

$$\mathbf{G}_u^{22} = -\mathbb{E}\left\{\frac{\partial^2 \mathcal{L}(\tilde{\mathbf{h}}_u \mid \tilde{\mathbf{h}}_{u-1})}{\partial \tilde{\mathbf{h}}_u \partial \tilde{\mathbf{h}}_u^H}\right\} - \mathbb{E}\left\{\frac{\partial^2 \mathcal{L}(\bar{\mathbf{y}}_{0,u}^{\text{td}} \mid \tilde{\mathbf{h}}_u)}{\partial \tilde{\mathbf{h}}_u \partial \tilde{\mathbf{h}}_u^H}\right\}. \tag{10}$$

Using the state and measurement model Equations in (48) and (50), respectively, the quantities $\mathcal{L}(\tilde{\mathbf{h}}_u \mid \tilde{\mathbf{h}}_{u-1})$ and $\mathcal{L}(\bar{\mathbf{y}}_{0,u}^{\text{td}} \mid \tilde{\mathbf{h}}_u)$ can be expressed as

$$\mathcal{L}(\tilde{\mathbf{h}}_u \mid \tilde{\mathbf{h}}_{u-1}) = \kappa_1 - \frac{(\tilde{\mathbf{h}}_u - \rho\tilde{\mathbf{h}}_{u-1})^H \widetilde{\boldsymbol{\Gamma}}^{-1} (\tilde{\mathbf{h}}_u - \rho\tilde{\mathbf{h}}_{u-1})}{1-\rho^2}, \tag{11}$$

$$\mathcal{L}(\bar{\mathbf{y}}_{0,u}^{\text{td}} \mid \tilde{\mathbf{h}}_u) = \kappa_2 - (\bar{\mathbf{y}}_{0,u}^{\text{td}} - \bar{\boldsymbol{\mathcal{D}}}\tilde{\mathbf{h}}_u)^H \tilde{\mathbf{R}}_\eta^{-1} (\bar{\mathbf{y}}_{0,u}^{\text{td}} - \bar{\boldsymbol{\mathcal{D}}}\tilde{\mathbf{h}}_u), \tag{12}$$

where the constants are $\kappa_1 = -N_r N_t L_h \log \pi (1-\rho^2) - \log\det(\boldsymbol{\Gamma})$, $\kappa_2 = -N_p N_r \log\pi - \log\det\left(\tilde{\mathbf{R}}_\eta\right)$ and $\widetilde{\boldsymbol{\Gamma}} = \left(\mathbf{I}_{N_r N_t} \otimes \boldsymbol{\Gamma}\right)$. Substituting the quantities $\mathcal{L}(\tilde{\mathbf{h}}_u \mid \tilde{\mathbf{h}}_{u-1})$ and $\mathcal{L}(\bar{\mathbf{y}}_{0,u}^{\text{td}} \mid \tilde{\mathbf{h}}_u)$ from (11), (12) into (9)-(10) determines the matrices $\mathbf{G}_u^{11}, \mathbf{G}_u^{12}, \mathbf{G}_u^{21}$ and $\mathbf{G}_u^{22}$. These are further substituted into (8), followed by further simplification to obtain the matrix $\mathbf{J}_u$ as

$$\mathbf{J}_u = \left(\rho^2 \mathbf{J}_{u-1} + \left(1-\rho^2\right)\widetilde{\boldsymbol{\Gamma}}\right)^{-1} + \bar{\boldsymbol{\mathcal{D}}}^H \tilde{\mathbf{R}}_\eta^{-1} \bar{\boldsymbol{\mathcal{D}}}. \tag{13}$$

The BCRLB of the MSE of the TD-GSBL-KF estimate is finally expressed as

$$\mathrm{MSE}\left(\tilde{\mathbf{h}}_u^{\text{TD-GSBL-KF}}\right) \triangleq \mathbb{E}\left\{\left\|\tilde{\mathbf{h}}_u^{\text{TD-GSBL-KF}} - \tilde{\mathbf{h}}_u\right\|_2^2\right\} \geq \mathrm{Tr}\{\mathbf{J}_u^{-1}\}. \tag{14}$$

When ρ is close to 1, one can verify that the time-recursive update equation of (13) becomes $\mathbf{J}_u \approx \mathbf{J}_{u-1} + \bar{\boldsymbol{\mathcal{D}}}^H \tilde{\mathbf{R}}_\eta^{-1} \bar{\boldsymbol{\mathcal{D}}}$, which further implies that $\mathbf{J}_u \succeq \mathbf{J}_{u-1}$. This leads to the decreasing MSE trend with respect to the time index u. These BCRLB trends can also be verified from our simulation results.

The BCRLB corresponding to the KFS framework is developed next. It follows from (13) that the forward recursion for the BFIM $\mathbf{J}_{u|u}$ is given by

$$\mathbf{J}_{u|u} = \mathbf{J}_{u|u-1} + \bar{\boldsymbol{\mathcal{D}}}^H \tilde{\mathbf{R}}_\eta^{-1} \bar{\boldsymbol{\mathcal{D}}}.$$

where we have $\mathbf{J}_{u|u-1} = \left(\rho^2 \mathbf{J}_{u-1|u-1}^{-1} + (1-\rho^2)\,\widetilde{\boldsymbol{\Gamma}}\right)^{-1}$. Employing the results derived in [2], the BFIM for the backward recursion becomes

$$\mathbf{J}_{u-1|U}^{-1} = \mathbf{J}_{u-1|u-1}^{-1} + \rho^2 \mathbf{J}_{u-1|u-1}^{-1} \mathbf{J}_{u|u-1} \left[\mathbf{J}_{u|U}^{-1} - \mathbf{J}_{u|u-1}^{-1}\right] \mathbf{J}_{u|u-1}^H \mathbf{J}_{u-1|u-1}^{-H}.$$

The BCRLB for the mean square error (MSE) of the time-domain TD-GSBL-KFS estimate is given by

$$\text{MSE}\left(\tilde{\mathbf{h}}_u^{\text{TD-GSBL-KFS}}\right) \triangleq \mathbb{E}\left\{ \left\| \tilde{\mathbf{h}}_u^{\text{TD-GSBL-KFS}} - \tilde{\mathbf{h}}_u \right\|_2^2 \right\} \geq \text{Tr}\left\{ \mathbf{J}_{u|U}^{-1} \right\}. \tag{15}$$

Both the BCRLBs of the IAM- and TD-based SISO-FBMC channel estimation using (16) and (24), as well as the MIMO-FBMC channel estimation using (21), and the doubly-selective channel estimation using (49), can be derived along similar lines. For example, for the IAM-based MIMO-FBMC system, the FIMs $\mathbf{J}_D$ and $\mathbf{J}_P$ can be expressed as $\mathbf{J}_D = \boldsymbol{\Phi}^H \bar{\mathbf{R}}_\eta^{-1} \boldsymbol{\Phi}$ and $\mathbf{J}_P = \left(\boldsymbol{\Gamma} \otimes \mathbf{I}_{N_r N_t}\right)^{-1}$. Thus, the Bayesian FIM $\mathbf{J}_B$ is expressed as $\mathbf{J}_B = \boldsymbol{\Phi}^H \bar{\mathbf{R}}_\eta^{-1} \boldsymbol{\Phi} + \left(\boldsymbol{\Gamma} \otimes \mathbf{I}_{N_r N_t}\right)^{-1}$. The MSE of the estimate of the vectorized channel $\bar{\mathbf{h}}$ can in turn be formulated as $\text{MSE}\left(\bar{\mathbf{h}}^{\text{IAM-BSBL}}\right) \geq \text{Tr}\left\{\mathbf{J}_B^{-1}\right\}$. Furthermore, for its doubly-selective extension, the BFIM $\mathbf{J}_u$ can be recursively computed as $\mathbf{J}_u = \left(\rho^2 \mathbf{J}_{u-1} + (1-\rho^2)\,\bar{\boldsymbol{\Gamma}}\right)^{-1} + \boldsymbol{\Phi}^H \bar{\mathbf{R}}_\eta^{-1} \boldsymbol{\Phi}$, where $\bar{\boldsymbol{\Gamma}} = \left(\boldsymbol{\Gamma} \otimes \mathbf{I}_{N_r N_t}\right)$. Finally, the BCRLB of the MSE of the IAM-BSBL-KF estimate is expressed as $\text{MSE}\left(\bar{\mathbf{h}}_u^{\text{IAM-BSBL-KF}}\right) \geq \text{Tr}\{\mathbf{J}_u^{-1}\}$.

V. BER Comparison of the Proposed and Existing Schemes

Fig. 3 shows the coded BER performance of MIMO-FBMC systems using channel estimates obtained from the proposed and existing schemes. A 2×2 MIMO system with $N = 64$ subcarriers is considered together with a rate $1/3$ LDPC code. A 3-tap equalizer, similar to [3], is used for data detection in the TD system, whereas a one-tap ZF equalizer is used for the IAM system. BER with perfect CSI (PCSI) has also been plotted for the reference. The TD system using GSBL can be seen to achieve the lowest BER, outperforming, viz. IAM with BSBL and both TD as well as IAM-based OMP and LS estimators. The NMSE floor of channel estimation in the IAM-based techniques naturally leads to the BER floor at high SNRs.

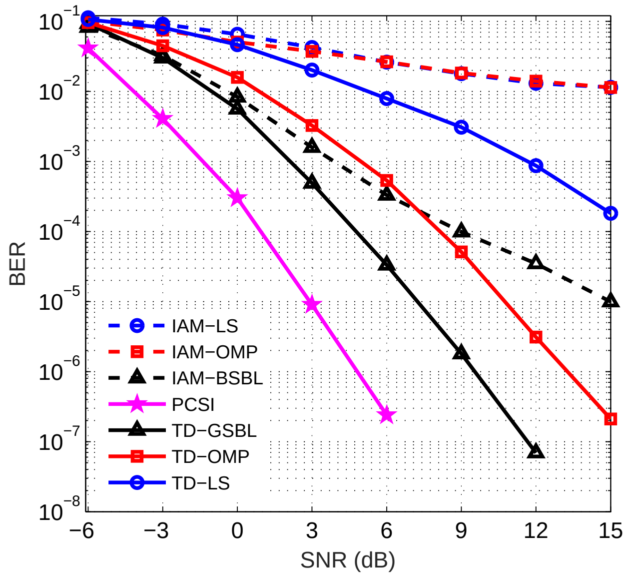

Fig. 3: BER of the proposed and existing methods for quasi-static sparse CSI estimation in 2×2 coded MIMO-FBMC systems with a rate $1/3$ low-density parity-check (LDPC) code, $N = 64$, $N_p = 28$, $z = 3$, BPSK modulation and $L_h = 16$ with 4 dominant paths.

References

[1] H. L. Van Trees and K. L. Bell, "Bayesian bounds for parameter estimation and nonlinear filtering/tracking," *AMC*, vol. 10, p. 12, 2007.

[2] N. Bergman, "Recursive bayesian estimation: Navigation and tracking applications," Ph.D. dissertation, Linköping University, 1999.

[3] A. Ikhlef and J. Louveaux, "Per subchannel equalization for MIMO FBMC/OQAM systems," in *2009 IEEE Pacific Rim Conference on Communications, Computers and Signal Processing*. IEEE, 2009, pp. 559–564.