Department of Mathematics and Statistics, I.I.T. Kanpur

MTH 101N - End Semester Examination -21.11.2007

Maximum Marks: 100 Time: 8:00-11:00 a.m.

- Please number the pages and answer all parts of the same question together.
- On the front page, against each question, indicate the page number on which it is answered.

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1. (a) Let $\{x_n\}$ be defined by $x_1 = 1, x_2 = 2$ and $x_n = \frac{1}{2}(x_{n-2} + x_{n-1}),\ \forall n \geq 3$. Show that the sequence $\{x_n\}$ converges. Find its limit. [6]

   (b) Discuss the convergence of the series $\sum_{1}^{\infty}(\ln n)\ \sin(\frac{1}{n^2})$. [6]

2. (a) Let $f : [0, 2] \to \mathbb{R}$ be a continuous function and $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ with $|x - y| = 1$ such that $f(x) = f(y)$. [6]

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. On $[a, b]$, define the average value of f by $f^*_{a,b} = \frac{1}{b-a}\int_a^b f(t)dt$. Using the fundamental theorem of calculus, show that there exists a point $s \in (0, 1)$ such that $f^*_{0,1} = f(s)$. [6]

3. (a) Let P_2 be a partition which contains two points more than a partition P_1, where P_1 and P_2 are both partitions of $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that $L(P_2, f) \leq U(P_1, f)$. [5]

   (b) Let $f : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Assume that f is bounded with a bounded second derivative. Let $P = \sup_{x\in\mathbb{R}} |f(x)|$, $Q = \sup_{x\in\mathbb{R}} |f''(x)|$, P and $Q \neq 0$. Show that $\sup_{x\in\mathbb{R}} |f'(x)| \leq 2\sqrt{PQ}$. [7]

4. (a) Find the volume of the solid obtained by rotating the region bounded by the curves $y = 1 - x^2$ and $y = 4 - 4x^2$ about the line $y = -1$. [6]

   (b) Sketch the curve $r = 1 + 2\cos\theta$. Find the total area of the region inside the curve. Set up the integral for the length of the curve. [8]

5. (a) Let

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

- Do all the directional derivatives of f exist at $(0,0)$?
- Is f continuous at $(0,0)$?
- Is f differentiable at $(0,0)$?

[7]

(b) Let $F(x,y) = (2x-y, x+2y)$ and D be the region outside the unit disk, above the curve $y = x^2 - 2$ and below the line $y = 2$. Verify Green's theorem. [8]

6. (a) Determine whether $\int\limits_0^\infty \frac{1-e^{-x}}{x\sqrt{x}}$ converges. [6]

(b) Let D be the region bounded by $x + y = 1$, $x = 0$ and $y = 0$. Evaluate $\iint\limits_D \cos(\frac{x-y}{x+y})\, dxdy$. [8]

7. Let $F = y\vec{i} + z\vec{j} + x\vec{k}$ be defined on the surface $S = \{(x,y,z) : x^2+y^2 = 1,\ 0 \leq z \leq x+2\}$ with an outward pointing normal. Verify Stokes Theorem. [10]

8. (a) Let $F(x,y,z) = \frac{1}{r^3}\vec{r}$, where $r = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$.
Let $D = \{(x,y,z) : 1 \leq x^2+y^2+z^2 \leq 9\}$. Verify the divergence theorem. [7]

(b) Without computing show that $\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}$. [4]