

Answer all the questions.

1. For what values of x does the series $\sum_2^{\infty} \frac{x^n}{n(\ln n)^{\frac{1}{2}}}$ converge? [6]

Solution :

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{n}{n+1} \right| \left| \left(\frac{\ln n}{\ln n+1} \right)^{\frac{1}{2}} \right| \rightarrow 1 \quad [2]$$

The radius of convergence is 1 and for $|x| < 1$, $\sum_2^{\infty} \frac{x^n}{n(\ln n)^{\frac{1}{2}}}$ converges.

Let $x = 1$. $\sum_2^{\infty} 2^k a_{2^k} = \sum_2^{\infty} 2^k \frac{1}{2^k \sqrt{k} (\ln 2)^{\frac{1}{2}}}$. Since this series diverges, by the Cauchy

condensation test $\sum_2^{\infty} \frac{1}{n(\ln n)^{\frac{1}{2}}}$ diverges. [2]

Let $x = -1$. Then $(a_n) = \left(\frac{1}{n(\ln n)^{\frac{1}{2}}} \right)$. Here, $a_n > 0$, a_n is decreasing and $a_n \rightarrow 0$.

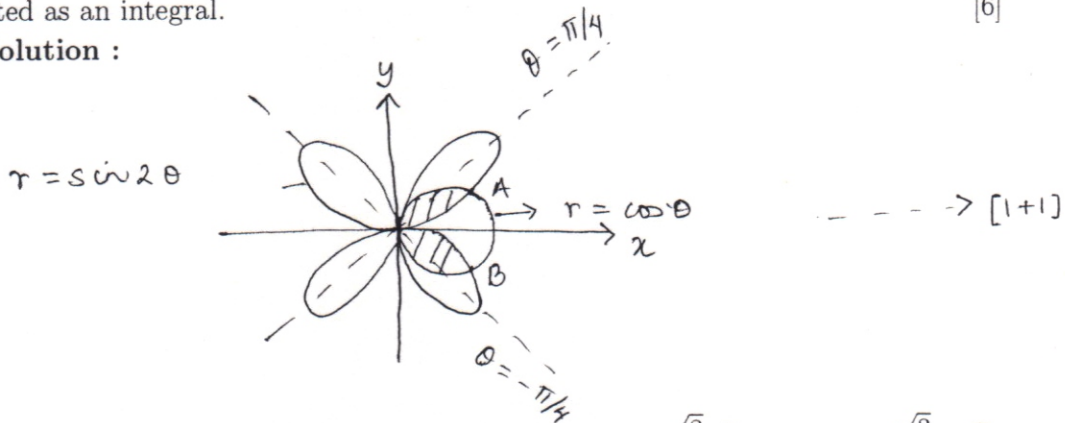
Thus, by the alternating series (Leibniz) test, $\sum_2^{\infty} \frac{(-1)^n}{n(\ln n)^{\frac{1}{2}}}$ converges. [2]

Hence for $-1 \leq x < 1$, the series $\sum_2^{\infty} \frac{x^n}{n(\ln n)^{\frac{1}{2}}}$ converges.

2. (a) Plot the curves $r = \sin 2\theta$ and $r = \cos \theta$. Consider the region that lies between both the curves. Express the area as an integral.

Now revolve this region around the x -axis. Express the area of the surface generated as an integral. [6]

Solution :



From the diagram, we see that the coordinates of $A = \left(\frac{\sqrt{3}}{2}, \frac{\pi}{6} \right)$ and $B = \left(\frac{\sqrt{3}}{2}, -\frac{\pi}{6} \right)$.

$$\text{Area of the shaded region} = 2 \frac{1}{2} \left(\int_0^{\frac{\pi}{6}} \sin^2 2\theta \, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2 \theta \, d\theta \right). \quad [2]$$

$$\text{Surface area} = \int_0^{\frac{\pi}{6}} 2\pi \sin \theta \sin 2\theta \sqrt{1 + 3 \cos^2 2\theta} \, d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \pi \sin 2\theta \, d\theta. \quad [2]$$

b. Does the integral $\int_0^1 \frac{\tan \sqrt{x}}{x + x^2} \, dx$ converge? Give reasons. [5]

Solution :

Since, for small x , $\tan \sqrt{x} \sim \sqrt{x}$, $\frac{\tan \sqrt{x}}{x + x^2} \sim \frac{\sqrt{x}}{x}$ (since for small x , x^2 is much smaller than x).

Do a limit comparison test with $\frac{1}{\sqrt{x}}$.

$$\text{Since } \lim_{x \rightarrow 0^+} \frac{\sqrt{x} \sin \sqrt{x}}{(x + x^2) \cos \sqrt{x}} = \lim_{x \rightarrow 0^+} \frac{\sin \sqrt{x}}{\sqrt{x}} \lim_{x \rightarrow 0^+} \frac{1}{(1 + x) \cos \sqrt{x}} = 1. \quad [3]$$

Since, $\int_0^1 \frac{1}{\sqrt{x}}$ is convergent, by the limit comparison test

$$\int_0^1 \frac{\tan \sqrt{x}}{x + x^2} \, dx \text{ is convergent.} \quad [2]$$

3 (a) Let

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Is f differentiable at $(0, 0)$? Supply the necessary arguments. [6]

Solution :

$$f \text{ is differentiable at } (0, 0) \implies \lim_{\|H\| \rightarrow 0} \frac{|f((0, 0) + H) - f(0, 0) - (f_x, f_y) \cdot H|}{\|H\|} = 0$$

$$\text{Now, } f_x(0, 0) = 0 = f_y(0, 0). \quad [2]$$

Therefore $(f_x, f_y) \cdot H = 0$, where $H = (h, k)$.

$$\text{Then, } \frac{|f(h, k) - f(0, 0) - \alpha \cdot H|}{\|H\|} = \frac{|hk(h^2 - k^2)|}{(h^2 + k^2)\sqrt{h^2 + k^2}}.$$

$$\frac{|f(h, k) - f(0, 0) - \alpha \cdot H|}{\|H\|} \leq \frac{\sqrt{h^2 + k^2} \sqrt{h^2 + k^2} (h^2 + k^2)}{(h^2 + k^2)\sqrt{h^2 + k^2}} = \sqrt{h^2 + k^2} \quad [2]$$

When $\|H\| \rightarrow 0$, then $\frac{|f(h, k) - f(0, 0) - \alpha \cdot H|}{\|H\|} \rightarrow 0$, thereby showing that f is differentiable at $(0, 0)$. [2]

- (b) A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at an angle of 60° at the center of the cylinder. Find the volume of the wedge. [5]

Solution :

$$A(x) = (x\sqrt{3})(2\sqrt{9-x^2}) \quad [3]$$

$$\text{Volume} = 2\sqrt{3} \int_0^3 x\sqrt{9-x^2} dx = 18\sqrt{3}. \quad [2]$$

- 4 (a) Let h be continuous and f, g be differentiable on \mathbb{R} . If

$$F(x) = \int_{f(x)}^{g(x)} h(t) dt,$$

then prove that F is differentiable. Find $F'(x)$. [6]

Solution :

$$F(x) = \int_a^{g(x)} h(t) dt + \int_{f(x)}^a h(t) dt = \int_a^{g(x)} h(t) dt - \int_a^{f(x)} h(t) dt = I_1 + I_2. \quad [2]$$

By the first fundamental theorem of calculus,

$$\text{we know that } \frac{d}{dx} \left(\int_0^x m(t) dt \right) = m(x) \quad [1]$$

By the chain rule $\frac{d}{dx} I_1 = h(g(x))g'(x)$.

Similarly, $\frac{d}{dx} I_2 = h(f(x))f'(x)$. [2]

Hence, $F'(x)$ exists and $F'(x) = h(g(x))g'(x) - h(f(x))f'(x)$. [1]

- (b) A particle moves along a circle of radius 3 with uniform speed t^2 , at time t . Compute the tangential and the normal components of the acceleration at time $t = 3$. Also, find the time required to complete the full circle. [6]

Solution :

$$\text{Let } a_T \text{ be the tangential component. } a_T = \frac{d^2 s}{dt^2} = 2t. \quad a_T = 6 \text{ when } t = 3. \quad [1]$$

$$\kappa \text{ for a circle of radius 3 is } \frac{1}{3}. \quad [1]$$

$$\text{Let } a_N \text{ be the normal component. Then } a_N = \frac{1}{3}t^4 = 27 \text{ when } t = 3. \quad [2]$$

$$\text{The length of the circle is } 6\pi. \quad s = \int_0^T t^2 dt = 6\pi. \quad T = (18\pi)^{\frac{1}{3}}. \quad [2]$$