Department of Mathematics and Statistics, I.I.T. Kanpur

MTH 101N - End Semester Examination -21.11.2007

Maximum Marks: 100 Time: 8:00-11:00 a.m.

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1. (a) Let $\{x_n\}$ be defined by $x_1 = 1, x_2 = 2$ and $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$, $\forall\, n \geq 3$. Show that the sequence $\{x_n\}$ converges. Find its limit. [6]

Solution : Clearly, $|x_{n+1} - x_n| = \frac{1}{2}|x_n - x_{n-1}|$. [2]
Here, we see that $\alpha = \frac{1}{2} < 1$. Hence, $\{x_n\}$ is a Cauchy sequence. Since, every Cauchy sequence is convergent, $\{x_n\}$ is a convergent sequence. [2]
Let (x_n) converge to l. Now, every subsequence of a convergent sequence is convergent and converges to the same limit. Consider the subsequence $\{x_{2n+1}\}$, $n \geq 0$.
By induction, we can see that $x_{2n+1} = 1 + \frac{1}{2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{2n-1}} = 1 + \frac{2}{3}(1 - \frac{1}{4^n})$.
Now, $x_{2n+1} \to \frac{5}{3}$. The limit, $l = \frac{5}{3}$. [2]

(b) Discuss the convergence of the series $\sum_1^\infty (\ln n)\, \sin(\frac{1}{n^2})$. [6]

Solution : Let $\sum a_n = \sum_1^\infty (\ln n)\, \sin(\frac{1}{n^2})$.
Since, $\sin\frac{1}{n^2} \sim \frac{1}{n^2}$ as $n \to \infty$, $(\ln n)\, \sin(\frac{1}{n^2}) \sim \frac{\ln n}{n^2}$.
Let $\sum_1^\infty b_n = \sum_1^\infty \frac{1}{n^{1+\alpha}}$, where $0 < \alpha < 1$. Here $\frac{(\ln n)\, \sin(\frac{1}{n^2})}{\frac{1}{n^{1+\alpha}}} \to 0$. [3]
By the limit comparison test, $\sum_1^\infty (\ln n)\, \sin(\frac{1}{n^2})$ converges,
since $\sum_0^\infty b_n$ converges. [3]

2. (a) Let $f : [0, 2] \to \mathbb{R}$ be a continuous function and $f(0) = f(2)$. Prove that there exist $x, y \in [0, 2]$ with $|x - y| = 1$ such that $f(x) = f(y)$. [6]

Solution : Define $g : [0, 1] \to \mathbb{R}$ by $g(x) = f(x + 1) - f(x)$. [2]
Clearly, g is continuous on $[0, 1]$. Now, $g(0) = f(1) - f(0)$, $g(1) = f(2) - f(1)$
$\implies g(1) + g(0) = f(2) - f(0) = 0$.
Hence, $g(0) = 0 = g(1)$ or $g(0) = -g(1) \neq 0$. [2]
If $g(0) = 0 = g(1)$, then $x = 1$, $y = 2$ and $f(1) = f(2)$. [1]
If $g(0) = -g(1) \neq 0$, by the intermediate value theorem, there exists $y \in (0, 1)$, such that $g(y) = 0$. Hence, $f(y + 1) = f(y)$. [1]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. On $[a, b]$, define the average value of f by $f^*_{a,b} = \frac{1}{b-a}\int\limits_a^b f(t)dt$. Using the fundamental theorem of calculus, show that there exists a point $s \in (0, 1)$ such that $f^*_{0,1} = f(s)$. [6]

Solution : Let $F(x) = \int\limits_a^x f(t)\,dt$. [1]

Since, f is continuous on $[a, b]$, by the first fundamental theorem of calculus, F is differentiable on $[a, b]$. [2]

By the mean value theorem, $\exists\, c \in (a, b)$, such that

$$F'(c) = \frac{F(b) - F(a)}{b-a} = \frac{1}{b-a}\int\limits_a^b f(t)\,dt = f^*_{a,b}. \quad [2]$$

Let $a = 0$ and $b = 1$. The result follows from the previous step. [1]

3. (a) Let P_2 be a partition which contains two points more than a partition P_1, where P_1 and P_2 are both partitions of $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Show that $L(P_2, f) \le U(P_1, f)$. [5]

Solution : Let $P_1 = \{a = x_0, x_1, \ldots x_n = b\}$ and
$P_2 = \{x_0, x_1, \ldots, x_i, y, x_{i+1}, \ldots, x_j, z, x_{j+1}, \ldots x_n\}$ be two partitions of $[a, b]$. [1+1]
Since, P_2 is finer than P_1, from the definition of the lower Riemann sum, we can show that $L(P_1, f) \le L(P_2, f)$.
Similarly, we show that, $U(P_2, f) \le U(P_1, f)$. [1]
For any partition P of $[a, b]$, $L(P, f) \le U(P, f)$. [1]
Hence, $L(P_1, f) \le L(P_2, f) \le U(P_2, f) \le U(P_1, f)$ [1]

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a thrice differentiable function. Assume that f is bounded with a bounded second derivative. Let $P = \sup\limits_{x\in\mathbb{R}} |f(x)|$, $Q = \sup\limits_{x\in\mathbb{R}} |f''(x)|$, P and $Q \ne 0$. Show that $\sup\limits_{x\in\mathbb{R}} |f'(x)| \le 2\sqrt{PQ}$. [7]

Solution : Let $x \in \mathbb{R}$ and $h > 0$.
By Taylor's theorem $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(c)$, where $c \in (x, x+h)$. [2]

$$f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2} f''(c). \quad [2]$$

$$|f'(x)| \le \frac{2P}{h} + \frac{hQ}{2}.$$

Choose $h = 2\sqrt{\frac{P}{Q}}$. [2]

4. (a) Find the volume of the solid obtained by rotating the region bounded by the curves $y = 1 - x^2$ and $y = 4 - 4x^2$ about the line $y = -1$. [6]

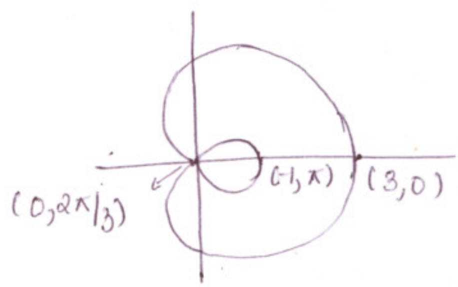
Solution : The curves intersect at the points $(-1, 0)$ and $(1, 0)$. [2]

By the washer method, the volume $V = \int_{-1}^{1} \pi((5 - 4x^2)^2 - (2 - x^2)^2)dx$ [2]

$V = 24\pi$. [2]

(b) Sketch the curve $r = 1 + 2\cos\theta$. Find the total area of the region inside the curve. Set up the integral for the length of the curve. [8]

Solution :



→ [3].

Area $A = 2\int_{0}^{\frac{2\pi}{3}} \frac{1}{2}(1 + 2\cos\theta)^2\, d\theta$ [2]

$A = 2\pi + \frac{3\sqrt{3}}{2}$. [1]

Length of the curve $s = 2\int_{0}^{\pi} \sqrt{5 + 4\cos\theta}d\theta$. [2]

5. (a) Let

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

- Do all the directional derivatives of f exist at $(0, 0)$?
- Is f continuous at $(0, 0)$?
- Is f differentiable at $(0, 0)$?

[7]

Solution : Let $u = (u_1, u_2)$ be a unit vector.
The directional derivative of f at $(0, 0)$ in the direction of u is

$$D_{(0,0)}u = \lim_{t \to 0} \frac{f(tu_1, tu_2) - f(0, 0)}{t} = \begin{cases} \frac{u_1^2}{u_2}, & u_2 \neq 0 \\ 0, & u_2 = 0. \end{cases}$$

[2]

Hence the directional derivative exists in every direction. [1]

Approaching $(0,0)$ along the path $y = x$, we see that

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{x\to 0} \frac{x^3}{x^2(x^2+1)} = 0.$$

Along the path $y = x^2$, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2} \neq 0.$

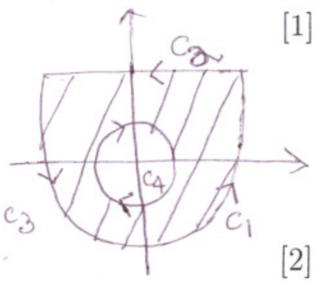
Hence, f is not continuous at $(0,0)$. [3]

Since f is not continuous at $(0,0)$, it is not differentiable at $(0,0)$. [1]

(b) Let $F(x,y) = (2x - y, x + 2y)$ and D be the region outside the unit disk, above the curve $y = x^2 - 2$ and below the line $y = 2$. Verify Green's theorem. [8]

Solution : Let $F(x,y) = (2x-y, x+2y)$. Then $M(x,y) = 2x-y$, $N(x,y) = x+2y$.
By Green's Theorem, $\int\int_R (M_x + N_y)\, dx\, dy = \int\limits_C M\, dy - N\, dx$. [1]

$div\, F = M_x + N_y = 4.$ [1]



Therefore, $\int\int_R M_x + N_y dx\, dy = 4$ area of $R = 4(A_1 - \pi)$

Here $A_1 = 2\int\limits_0^2 (2 - (x^2 - 2))\, dx = \frac{32}{3}.$

Hence $\displaystyle\int\int_R M_x + N_y dx\, dy = \frac{128}{3} - 4\pi$ [2]

On C_1, $y = x^2 - 2$, $\displaystyle\int\limits_{C_1} M\, dy - N\, dx = \int\limits_0^2 (2x^2 - 2x^3 + 3x + 4)\, dx = \frac{34}{3}$ [1]

On C_2, $y = 2$, $\displaystyle\int\limits_{C_2} M\, dy - N\, dx = -\int\limits_2^{-2} (x + y)\, dx = 16.$ [1]

On C_3, $y = x^2 - 2$, $\displaystyle\int\limits_{C_3} M\, dy - N\, dx = \int\limits_{-2}^0 (2x^2 - 2x^3 + 3x + 4)\, dx = \frac{46}{3}.$ [1]

On C_4, $x = \cos\theta$ and $y = \sin\theta$. $M dx - N dy = 2 d\theta$.

$$\int\limits_{C_4} M\, dy - N\, dx = 2\int\limits_{2\pi}^{0} d\theta = -4\pi.$$ [1]

The total line integral $= \int\limits_{C_1} + \int\limits_{C_2} + \int\limits_{C_3} + \int\limits_{C_4} = \frac{128}{3} - 4\pi.$

This verifies Green's Theorem.

6. (a) Determine whether $\int_0^\infty \frac{1-e^{-x}}{x\sqrt{x}}\,dx$ converges. [6]

Solution : Let $I = \int_0^\infty \frac{1-e^{-x}}{x\sqrt{x}}\,dx = \int_0^1 \frac{1-e^{-x}}{x\sqrt{x}}\,dx + \int_1^\infty \frac{1-e^{-x}}{x\sqrt{x}}\,dx = I_1 + I_2.$

For the integral I_1. Let $g(x) = \frac{1}{\sqrt{x}}$. [1]

Then $\lim_{x\to 0} = \frac{\frac{1-e^{-x}}{x\sqrt{x}}}{g(x)} = 1.$

By the limit comparison test, I_1 converges if and only if $\int_0^1 g(x)\,dx$ converges.

Since, $\int_0^1 g(x)\,dx$ converges, so does I_1. [2]

For the integral I_2. Let $h(x) = \frac{1}{x\sqrt{x}}$. [1]

Then $\lim_{x\to\infty} = \frac{\frac{1-e^{-x}}{x\sqrt{x}}}{h(x)} = 1.$

By the limit comparison test, I_2 converges if and only if $\int_0^1 g(x)\,dx$ converges.

Since, $\int_0^1 g(x)\,dx$ converges, so does I_2. [2]

Since, both I_1 and I_2 converge, so does $I = I_1 + I_2$.

(b) Let D be the region bounded by $x + y = 1$, $x = 0$ and $y = 0$. Evaluate $\int\int_D \cos(\frac{x-y}{x+y})\,dxdy.$ [8]

Solution : Let $u = x + y$ and $v = x - y$. [1]
The jacobian matrix has determinant $-\frac{1}{2}$. [2]
The limits of integration for u and v are $0 \leq u \leq 1$ and $-u \leq v \leq u$. [2]
Using the change of variable formula, we see that

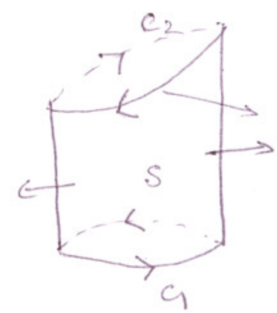
$$\int\int_D \cos(\frac{x-y}{x+y})\,dxdy = \int_0^1 \int_{-u}^{u} \cos(\frac{v}{u})\,\frac{1}{2}dvdu \quad [2]$$

$$\int\int_D \cos(\frac{x-y}{x+y})\,dxdy = \frac{\sin 1}{2}. \quad [1]$$

7. Let $F = y\vec{i} + z\vec{j} + x\vec{k}$ be defined on the surface $S = \{(x, y, z) : x^2+y^2 = 1,\ 0 \leq z \leq x+2\}$ with an outward pointing normal. Verify Stokes Theorem. [10]

Solution : Stoke's theorem states that $\int\int_S \nabla \times F.n\,d\sigma = \int_{C=\partial S} F.dr$ [1]

$r(\theta, t) = \{(\cos\theta, \sin\theta, t) : 0 \leq \theta < 2\pi,\ 0 \leq t \leq 2 + \cos\theta\}$ is a parametrisation of S. [1]



$\nabla \times F = (-1, -1, -1)$. [1]

$n d\sigma = (\cos\theta, \sin\theta, 0) d\theta dt$. [1]

Hence, $\int\int_S \nabla \times F.n\, d\sigma = -\pi$. [1]

C_1 and C_2 are the boundary circle of the surface.

Thus $\int_{\partial S} F.dr = \int_{C_1} F.dr - \int_{-C_2} F.dr$ [1]

On C_1, $r(t) = (\cos\theta, \sin\theta, 0)$ and $\int_{C_1} F.dr = -\pi$. [2]

On C_2, $r(t) = (\cos\theta, \sin\theta, 2 + \cos\theta)$ and $\int_{C_2} F.dr = 0$. [2]

Since $\int\int_S \nabla \times F.n\, d\sigma = -\pi = \int_{\partial S} F.dr$, Stokes theorem is verified.

8. (a) Let $F(x, y, z) = \frac{1}{r^3}\vec{r}$, where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$.
Let $D = \{(x, y, z) : 1 \leq x^2 + y^2 + z^2 \leq 9\}$. Verify the divergence theorem. [7]

Solution : The divergence theorem states that $\int\int_{\partial D} F.n d\sigma = \int\int\int_D div\, F dV$. [1]
$\partial D = S_1 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \cup S_2 = \{(x, y, z) : x^2 + y^2 + z^2 = 9\}$.
The unit normal to each of these surfaces points outward from the origin. [1]
Since $div\, F = 0$, the volume integral is 0. [1]
The normal $n = \frac{\vec{r}}{r}$ in both the cases. [2]
$\int\int_{\partial D} F.n d\sigma = \int\int_{S_2} F.n d\sigma - \int\int_{S_1} F.n d\sigma$ [1]
Hence $\int\int_{S_2} F.n d\sigma - \int\int_{S_1} F.n d\sigma = \frac{1}{9} 36\pi - 4\pi = 0$. [1]
Hence, the divergence theorem is verified.

(b) Without computing show that $\frac{1}{9} < \sqrt{66} - 8 < \frac{1}{8}$. [4]

Solution : Define $f : [64, 66] \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. [1]
Clearly, f is both continuous and differentiable on $[64, 66]$.
By the mean value theorem, $\frac{f(66) - f(64)}{66 - 64} = f'(c)$, where $c \in (64, 66)$.
Hence, $\frac{\sqrt{66} - 8}{2} = \frac{1}{2\sqrt{c}}$. [2]
$\frac{1}{9} = \frac{1}{\sqrt{81}} < \frac{1}{\sqrt{66}} < \sqrt{66} - 8 < \frac{1}{\sqrt{64}} = \frac{1}{8}$. [1]