1 (a) The asymptotes are $y = x - 2$ and $x = -2$. [1]
$f'(x) = \frac{x^2+4x+1}{(x+2)^2}$. $f'(x) = 0 \Rightarrow x = -2 \pm \sqrt{3}$ [1]

$f''(x) > 0$ if $x = -2 + \sqrt{3}$ and $f''(x) < 0$ if $x = -2 - \sqrt{3}$. Therefore f has a local maximum at $-2 - \sqrt{3}$ and a local minimum at $-2 + \sqrt{3}$. [1]

Since $f''(x) > 0$ if $x > -2$, it is convex and $f''(x) < 0$ if $x < -2$, it is concave. [1]
Graph (refer to a separate file) [1]

(b) By induction, show that $x_n \leq 1$, for each n. [2]
$|x_{n+2} - x_{n+1}| = \frac{1}{7}|x_{n+1}^2 - x_n^2| \leq \frac{2}{7}|x_{n+1} - x_n|$. Therefore, (x_n) is a cauchy sequence and hence convergent. [2]

$l = \frac{7-\sqrt{41}}{2}$ (since $x_n \leq 1$, $l \leq 1$. [1]

2 (a) $0 \leq \frac{\sin^3(n+1)+2}{2^n+n^2} < \frac{3}{2^n}$ [2]
$\sum_{n=1}^{\infty} \frac{1}{2^n}$ is convergent. [2]
By comparison test the given series is convergent. [1]
(Alternately, instead of $\frac{3}{2^n}$, you can use $\frac{3}{n^2}$.)

(b) Let $f(t) = e^t$. [2]
By the MVT on $[0, x]$, we see $1 = e^0 < \frac{e^x - e^0}{x - 0} = e^c < e^x$, for some $c \in (0, x)$. [2]
The result follows by taking logarithms. [1]

3 (a) First, let $x \in \mathbb{R} \setminus \mathbb{Z}$. $\exists\, n \in \mathbb{Z}$, such that $n < x < n+1$. Here, $f(x) = x - n$, which is a sum of continuous functions and is therefore continuous. [2]

Now, let $m_0 \in \mathbb{Z}$. Now, $\lim_{x \to m_0^-} f(x) = 1$ and $\lim_{x \to m_0^+} f(x) = 0$. [2]
Therefore, f is discontinuous at every integer point. [1]

(b) Let $f(x) = x^2 - x\sin x - \cos x$. $f'(x) = x(2 - \cos x)$.
$f'(x) = 0$ only when $x = 0$. [1]
Therefore, by Rolle's theorem, $f(x) = 0$ has atmost 2 solutions. [2]
Now, $f(-\frac{\pi}{2}) > 0$, $f(0) < 0$ and $f(\frac{\pi}{2}) > 0$. [1]
By the intermediate value property $f(x) = 0$ has exactly 2 solutions. [1]

4 (a) By Taylor's Theorem, $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(c)$, where $c \in (x, x+h)$ [2]
$\exists M \in \mathbb{R}$, such that $|f''(x)| < M$, $\forall\, x$ and for each $\epsilon > 0$, $\exists x_0$, such that $|f(x)| < \epsilon$, $\forall\, x > x_0$. [1]
Let $x > x_0$ and $h > 0$. Then $|f'(x)| < \frac{2\epsilon}{h} + \frac{h}{2}M$. [1]
Choose $h = 2\sqrt{\frac{\epsilon}{M}}$. [1]

(b) Since $f(c)$ is a local maximum, there exists a $\delta > 0$, such that
$\forall\ x \in (c - \delta, c + \delta),\ f(x) \leq f(c)$. [1]

$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0.$ [1]

$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0.$ [1]

Therefore $f'(c) = 0$. [1]

If c is an endpoint $f'(c)$ need not be 0. *Eg.* $f(x) = x$ on $[0, 1]$. [1]