Infinite Series

Series : Let (a_n) be a sequence of real numbers. Then the expression of the form

$$a_1 + a_2 + a_3 + ...$$

denoted by $\sum_{n=1}^{\infty} a_n$, is called a series.

Examples :

1. $1 + \frac{1}{2} + \frac{1}{3} + ...$ or $\sum_{n=1}^{\infty} \frac{1}{n}$
2. $1 + \frac{1}{4} + \frac{1}{9} + ...$ or $\sum_{n=1}^{\infty} \frac{1}{n^2}$

Partial sums : $S_n = a_1 + a_2 + a_3 + ... + a_n$ is called the nth partial sum of the series $\sum_{n=1}^{\infty} a_n$,

Convergence or Divergence of $\sum_{n=1}^{\infty} a_n$:

If $S_n \to S$ for some S then we say that the series $\sum_{n=1}^{\infty} a_n$ converges to S. If (S_n) does not converge then we say that the series $\sum_{n=1}^{\infty} a_n$ diverges.

Examples :

1. $\sum_{n=1}^{\infty} log(\frac{n+1}{n})$ diverges because $S_n = log(n+1)$.
2. $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges because $S_n = 1 - \frac{1}{n+1} \to 1$.
3. If $0 < x < 1$, then the geometric series $\sum_{n=0}^{\infty} x^n$ converges to $\frac{1}{1-x}$ because $S_n = \frac{1-x^{n+1}}{1-x}$.

Necessary condition for convergence : If $\sum_{n=1}^{\infty} a_n$ converges then $a_n \to 0$.

Proof : $S_{n+1} - S_n = a_{n+1} \to S - S = 0$.

Examples :

1. If $\mid x \mid \geq 1$, then $\sum_{n=1}^{\infty} x^n$ diverges because $a_n \nrightarrow 0$.
2. $\sum_{n=1}^{\infty} sinn$ diverges because $a_n \nrightarrow 0$.
3. $\sum_{n=1}^{\infty} log(\frac{n+1}{n})$ diverges, however, $log(\frac{n+1}{n}) \to 0$.

Necessary and sufficient condition for convergence : Suppose $a_n \geq 0 \ \forall \ n$. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if (S_n) is bounded above.

Example : The Harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges because $S_{2^k} \geq 1 + \frac{1}{2} + 2 \cdot \frac{1}{4} + 4 \cdot \frac{1}{8} + ... + 2^{k-1} \cdot \frac{1}{2^k} = 1 + \frac{k}{2} \ \forall \ k$.

Theorem : If $\sum_{n=1}^{\infty} \mid a_n \mid$ converges then $\sum_{n=1}^{\infty} a_n$ converges.

Proof : Since $\sum_{n=1}^{\infty} \mid a_n \mid$ converges the sequence of partial sums of $\sum_{n=1}^{\infty} \mid a_n \mid$ is Cauchy. Therefore, the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ is Cauchy.

Remark : Note that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=p}^{\infty} a_n$converges for any $p \geq 1$.

Let us determine the convergence or the divergence of a series by comparing it to one whose behavior is already known.

Comparison test : Suppose $0 \leqslant x_n \leqslant y_n$ for $n \geqslant k$ for some k. Then

(1) The convergence of $\sum_{n=1}^{\infty} y_n$ implies the convergence of $\sum_{n=1}^{\infty} x_n$.

(2) The divergence of $\sum_{n=1}^{\infty} x_n$ implies the divergence of $\sum_{n=1}^{\infty} y_n$.

Proof :

(1) Note that the sequence of partial sums of $\sum_{n=1}^{\infty} x_n$ is bounded.

(2) This statement is the contrapositive of (1).

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges because $\frac{1}{(n+1)(n+1)} \leqslant \frac{1}{n(n+1)}$. This implies that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

2. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because $\frac{1}{n} \leqslant \frac{1}{\sqrt{n}}$.

3. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $n^2 < n!$ for $n \geqslant 4$.

Cauchy Test or Cauchy condensation test: If $a_n \geq 0$ and $a_{n+1} \leq a_n \; \forall \; n$, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.

Proof : Let $S_n = a_1 + a_2 + ... + a_n$ and $T_k = a_1 + 2a_2 + ... + 2^k a_{2^k}$.

Suppose (T_k) converges. For a fixed n, choose k such that $2^k \geq n$. Then

$$\begin{aligned} S_n &= a_1 + a_2 + ... + a_n \\ &\leq a_1 + (a_2 + a_3) + ... + (a_{2^k} + ... + a_{2^{k+1}-1}) \\ &\leq a_1 + 2a_2 + ... + 2^k a_{2^k} \\ &= T_k. \end{aligned}$$

This shows that (S_n) is bounded above; hence (S_n) converges.

Suppose (S_n) converges. For a fixed k, choose n such that $n \geq 2^k$. Then

$$\begin{aligned} S_n &= a_1 + a_2 + ... + a_n \\ &\geq a_1 + a_2 + (a_3 + a_4)..... + (a_{2^{k-1}+1} + ... + a_{2^k}) \\ &\geq \tfrac{1}{2}a_1 + a_2 + 2a_4 + ... + 2^{k-1} a_{2^k} \\ &= \tfrac{1}{2}T_k. \end{aligned}$$

This shows that (T_k) is bounded above; hence (T_k) converges.

Examples:

1. $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

2. $\sum_{n=1}^{\infty} \frac{1}{n(logn)^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

3. If $a_n \geq 0, a_{n+1} \leq a_n \; \forall \; n$ and $\sum a_n$ converges then by Cauchy condensation test $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. Therefore $2^k a_{2^k} \to 0$, i.e., $2^{k+1} a_{2^k} \to 0$ as $k \to \infty$ which implies that $na_n \to 0$ as $n \to \infty$

Ratio test: Consider the series $\sum_{n=1}^{\infty} a_n, \; a_n \neq 0 \; \forall \; n$.

1. If $\mid \frac{a_{n+1}}{a_n} \mid \leq q$ eventually for some $0 < q < 1$, then $\sum_{n=1}^{\infty} \mid a_n \mid$ converges.
2. If $\mid \frac{a_{n+1}}{a_n} \mid \geq 1$ eventually then $\sum_{n=1}^{\infty} a_n$ diverges.

Proof:

1. Note that for some N, $\mid a_{n+1} \mid \leq q \mid a_n \mid \ \forall\, n \ \geq N$. Therefore, $\mid a_{N+p} \mid \leq q^p \mid a_N \mid$ $\forall\, p > 0$. Apply the comparison test.
2. In this case $\mid a_n \mid \not\to 0$.

Corollary : Suppose $a_n \ \neq 0 \ \forall\, n$, and $\mid \frac{a_{n+1}}{a_n} \mid \to L$ for some L.

1. If $L < 1$ then $\sum_{n=1}^{\infty} \mid a_n \mid$ converges.
2. If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$ we cannot make any conclusion.

Proof :

1. Note that $\mid \frac{a_{n+1}}{a_n} \mid < L + \frac{(1-L)}{2}$ eventually. Apply the previous theorem.
2. Note that $\mid \frac{a_{n+1}}{a_n} \mid > L - \frac{(L-1)}{2}$ eventually. Apply the previous theorem.

Examples :

1. $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $\frac{a_{n+1}}{a_n} \ \to 0$.
2. $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ diverges because $\frac{a_{n+1}}{a_n} = (1 + \frac{1}{n})^n \ \to e > 1$.
3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, however, in both these cases $\frac{a_{n+1}}{a_n} \ \to 1$.

Root Test : If $0 \ \leq a_n \ \leq x^n$ or $0 \ \leq {a_n}^{1/n} \ \leq x$ eventually for some $0 < x < 1$ then $\sum_{n=1}^{\infty} \mid a_n \mid$ converges.

Proof : Immediate from the comparison test.

Corollary: Suppose $\mid a_n \mid^{1/n} \to L$ for some L. Then

1. If $L < 1$ then $\sum_{n=1}^{\infty} \mid a_n \mid$ converges.
2. If $L > 1$ then $\sum_{n=1}^{\infty} a_n$ diverges.
3. If $L = 1$ we cannot make any conclusion.

Examples :

1. $\sum_{n=2}^{\infty} \frac{1}{(logn)^n}$ converges because $a_n^{1/n} = \frac{1}{logn} \ \to 0$.
2. $\sum_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$ converges because $a_n^{1/n} = \frac{1}{(1+\frac{1}{n})^n} \ \to \frac{1}{e} < 1$.
3. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, however, in both these cases $a_n^{1/n} \to 1$.

Leibniz test : If (a_n) is decreasing and $a_n \to 0$, then $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

Proof : Note that (S_{2n}) increases and (S_{2n+1}) decreases and both are bounded. Therefore both converge. Since $S_{2n+1} - S_{2n} = a_{2n+1} \to 0$, both (S_{2n+1}) and (S_{2n}) converge to the same limit and therefore (S_n) converges.

Examples : $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$, $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$ and $\sum_{n=2}^{\infty}(-1)^n \frac{1}{logn}$ converge.