Sequence

Informally, a sequence is a set with an order in the sense that there is a first element, second and so on. In other words to each positive integer 1,2,3, ..., we associate a real number. In the sequel, we will consider only sequences of real numbers.

Definition A function $f : \{1, 2, 3, \ldots\} \to I\!R$ is called a sequence of real numbers. We write $f(n) = x_n$ and then the sequence is denoted by $x_1, x_2, \ldots$, or simply by $\{x_n\}$. We call x_n the nth element of the sequence or the value of the sequence at n.

Some examples of sequences:

1. $\{n\} = 1, 2, 3, \ldots$
2. $\{\frac{1}{n}\} = 1, \frac{1}{2}, \frac{1}{3}, \ldots$
3. $\{\frac{(-1)^n}{n}\} = -1, \frac{1}{2}, \frac{-1}{3}, \ldots$
4. $\{1 - \frac{1}{n}\} = 0, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \ldots$
5. $\{1 + \frac{1}{10^n}\} = 1.1, 1.01, 1.001, \ldots$
6. $\{(-1)^n\} = -1, +1, -1, +1, \ldots$
7. $\{5\} = 5, 5, 5, \ldots$
8. $\{\frac{1+(-1)^n}{2}\} = 0, 1, 0, 1, 0, \ldots$

Before giving the formal definition of convergence of a sequence, let us take a look at the behaviour of the sequences in the above examples.

The elements of the sequences $\{\frac{1}{n}\}$, $\{1 - 1/n\}$ and $\{1 + 1/10^n\}$ seem to "approach" a single point as n increases. In these sequences the values are either increasing or decreasing as n increases, but they "eventually approach" a single point. Though the elements of the sequence $\{(-1)^n/n\}$ oscillate, they "eventually approach" the single point 0. Similarly for the sequence $5, 5, \ldots$. The common feature of these sequences is that the terms of each sequence "accumulate" at only one point. On the other hand, values of the sequence $\{n\}$ become larger and larger and do not accumulate anywhere. The elements of the sequence $\{(-1)^n\}$ oscillate between two different points -1 and 1; i.e., the elements of the sequence come close to -1 and 1 "frequently" as n increases.

2.1 Convergence of a Sequence

Let us distinguish sequences whose elements approach a single point as n increases (in this case we say that they converge) from those sequences whose elements do not. Geometrically it is clear that if the elements of the sequence $\{x_n\}$ come eventually inside *every* neighbourhood $(x_0 - \epsilon, x_0 + \epsilon)$, $\epsilon > 0$, of x_0, then $\{x_n\}$ approaches x_0.

Let us now state the formal definition of convergence.

We say that a sequence $\{x_n\}$ converges if there exist $x \in I\!R$ such that for every $\epsilon > 0$, there exists a positive integer N (depending on ϵ) such that $|x_n - x| < \epsilon$ for all $n \geq N$. It can be easily verified that if such a number x exists then it is unique. In this case, we say that the sequence $\{x_n\}$ converges to x and we call x the limit of the sequence $\{x_n\}$.

If x is the limit of $\{x_n\}$, we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.

Let us show that the sequence $\{\frac{1}{n}\}$ in Example (ii) has limit equal to 0. For arbitrary $\epsilon > 0$, the inequality

$$|x_n| = \frac{1}{n} < \epsilon$$

is true for all $n > \frac{1}{\epsilon}$ and hence for all $n > N$ where N is any natural number such that $N > \frac{1}{\epsilon}$. Thus for any $\epsilon > 0$, there is a natural number N such that $|x_n| < \epsilon$ for every $n \geq N$.

Similarly, we can prove that the sequence in Example (iii) converges to 0.

The sequence in Example (iv) converges to 1, because in this case

$$|1 - x_n| = |1 - \frac{n-1}{n}| = \frac{1}{n} \leq \epsilon$$

for all $n > N$ where N is any natural number greater than $\frac{1}{\epsilon}$.

Arithmetic Operations on Sequences

Since sequences of real numbers are real valued functions, the definition of sum, difference, product and quotient of sequences follows from the definition of the respective operations of real numbers. For example, if $\{x_n\}$ and $\{y_n\}$ are given then $\{x_n\} + \{y_n\}$ is the sequence $\{x_n + y_n\}$.

Theorem 2.1 Suppose $x_n \to x$ and $y_n \to y$. Then

1. $x_n + y_n \to x + y$
2. $x_n y_n \to xy$
3. $\frac{x_n}{y_n} \to \frac{x}{y}$ if $y \neq 0$ and $y_n \neq 0 \; \forall \; n$.

Proof: Exercise.

2.3 Monotone Sequences

We now give a simple criterion for the convergence of a sequence (without having any knowledge of its limit).

Suppose a sequence $\{x_n\}$ converges to x. Then, for $\epsilon = 1$, there exist N such that $|x_n - x| \leq 1 \; \forall \; n \geq N$. This implies $|x_n| \leq |x| + 1 \; \forall \; n \geq N$. If we let $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|\}$, then $|x_n| \leq M + |x| + 1 \; \forall \; n$. Hence $\{x_n\}$ is a bounded sequence.

Every convergent sequence is a bounded sequence. However, bounded sequences may not converge. We now find a condition on a bounded sequence which ensures the convergence of the sequence.

2.2 Definition We say that a sequence $\{x_n\}$ is increasing if $x_n \leq x_{n+1} \; \forall \; n$ and strictly increasing if $x_n < x_{n+1} \forall \; n$. Similarly, we define decreasing and strictly decreasing sequences.

The following result is an application of the least upper bound property of the real number system.

Theorem 2.2 Suppose $\{x_n\}$ is a bounded and increasing sequence. Then the least upper bound of the set $\{x_n : n \in N\}$ is the limit of $\{x_n\}$.

Proof: Suppose $\sup\limits_n x_n = M$. For given $\epsilon > 0$, there exists n_0 such that $M - \epsilon \leq x_{n_0}$. Since $\{x_n\}$ is increasing, we have $x_{n_0} \leq x_n \ \forall\ n \geq n_0$. This implies that $M - \epsilon \leq x_n \leq M \leq M + \epsilon \ \forall\ n \geq n_0$. That is $x_n \to M$. $\Box$

Exercise: Prove that a bounded decreasing sequence converges to $\inf\{x_n :\ n \in N\}$.

We will now give applications of the previous theorem.

The following theorem, called nested interval theorem, is a famous result about the set of real numbers.

Theorem 2.3 For each n, let $I_n = [a_n, b_n]$ be a (nonempty) bounded interval of real numbers such that

$$I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$

and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$. Then $\bigcap\limits_{n=1}^{\infty} I_n$ contains only one point.

Proof: Note that the sequences (a_n) and (b_n) are respectively increasing and decreasing sequences; moreover both are bounded. Hence both converge, say $a_n \to a$ and $b_n \to b$. Then $a_n \leq a$ and $b \leq b_n \ \forall\ n \in N$. Since $b - a = \lim(b_n - a_n) = 0$. Thus $a = b$. Since $a_n \leq b_n \ \forall\ n$; this implies that $a \in \bigcap\limits_{n=1}^{\infty} I_n$. Clearly if $x \neq a$ then x does not belong to $\bigcap\limits_{n=1}^{\infty} I_n$.

Example: The sequence $\{\left(1 + \frac{1}{n}\right)^n\}$ is convergent.

Proof: Let $x_n = \left(1 + \frac{1}{n}\right)^n$. By the binomial theorem

$$\begin{aligned}
x_n &= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{1\cdot 2} \cdot \frac{1}{n^2} + \cdots \frac{n(n-1)\cdots 1}{1\cdots 2\cdots n} \cdot \frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \\
&+ \cdots + \frac{1}{n!}(1 - \frac{1}{n})\cdots(1 - \frac{n-1}{n}) \\
&\leq 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n+1}) + \frac{1}{3!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1}) \\
&+ \cdots + \frac{1}{(n+1)!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1})\cdots(1 - \frac{n}{n+1}) \\
&= x_{n+1}
\end{aligned}$$

Therefore $\{x_n\}$ is an increasing sequence. Moreover,

$$\begin{aligned}
x_n &\leq 2 + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!} \\
&\leq 2 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}} \leq 3.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded and increasing. Hence by the previous theorem $\{x_n\}$ converges.

It is clear that the limit lies between 2 and 3. This limit is called the number e. Thus $\lim\limits_{n\to\infty}(1+\frac{1}{n})^n = e$.

2.4 Cauchy Sequences

We have seen one criterion for proving that a sequence converges without knowing its limit. We will now present another criterion.

Suppose that a sequence $\{x_n\}$ converges to x. Then for $\epsilon > 0$, there exists an N such that $|x_n - x| < \epsilon/2 \ \forall\ n \geq N$. Hence for $n, m \geq N$ we have $|x_n - x_m| = |x_n - x + x - x_m| \leq |x_n - x| + |x - x_m| < \epsilon$. Thus we arrive at the following conclusion.

If a sequence $\{x_n\}$ converges then it satisfies the Cauchy's condition: for $\epsilon > 0$, there exist N such that $|x_n - x_m| < \epsilon \ \forall\ n, m \geq N$.

If a sequence $\{x_n\}$ satisfies the Cauchy's condition then we say that $\{x_n\}$ is a Cauchy sequence.

If a sequence converges then the elements of the sequence get close to the limit as n increases. In case of Cauchy sequence the elements get close to each other as m, n increases.

Exercise : Verify that every Cauchy sequence is a bounded sequence.

Thus a Cauchy sequence is a bounded sequence with some additional property. Moreover, intuitively it seems as if it converges. So it is natural to ask whether every Cauchy sequence does converge.

Theorem 2.4 Every Cauchy sequence is convergent.

Proof: Since every Cauchy sequence is bounded, for an arbitrary n, we define

$$a_n = \inf_{k>n} x_k \quad \text{and} \quad b_n = \sup_{k>n} x_k.$$

Then the intervals $[a_n, b_n]$ form a nested family:

$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n].$$

Since $\{x_n\}$ is a Cauchy sequence, $b_n - a_n \to 0$ as $n \to \infty$. Hence by the nested interval theorem, there exists x such that $x \in \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$. This x is the limit of $\{x_n\}$ (verify !).

2.5 Subsequences

Let $\{x_n\}$ be a sequence and let $\{n_k\}$ be any sequence of positive integers such that $n_1 < n_2 < n_3 < \ldots$. The sequence $\{x_{n_k}\}$ is called a subsequence. Note that here k varies from 1 to ∞.

A subsequence is formed by deleting some of the elements of the sequence and retaining the remaining with the same order.

For example, $\{\frac{1}{k^2}\}$ and $\{\frac{1}{2^k}\}$ (k varies from 1 to ∞) are subsequences of $\{\frac{1}{n}\}$, where $n_k = k^2$ and $n_k = 2^k$. Sequences $(1,1,1,\ldots)$ and $(0,0,0,\ldots)$ are both subsequences of $(1,0,1,0,\ldots)$. We also note that every sequence is a subsequence of itself.

The following three results are easy to verify so they are left as exercise.

Theorem 2.5 Suppose $x_n \to x$. Then every subsequence of $\{x_n\}$ also converges to x.

Exercise : If $\{x_n\}$ is a Cauchy sequence of real numbers which has a subsequence converging to x, show that $\{x_n\}$ itself converges to x.

From the previous examples it is clear that a given sequence may have convergent subsequence though the sequence itself is not convergent. We also note that if all convergent subsequences of a sequence converge to the same limit even then the original sequence may not converge. At the same time, we have the following result.

Theorem 2.6 If every subsequence of a sequence $\{x_n\}$ has a convergent subsequence which converges to the same limit then the original sequence converges.

Proof : Exercise.

We say that x is a cluster point of $\{x_n\}$ if there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x$.

Theorem 2.7 An element x is a cluster point of a sequence $\{x_n\}$ if and only if for every $\epsilon > 0$, there exist infinitely many integers n such that $|x_n - x| < \epsilon$.

If x is a cluster point of a sequence $\{x_n\}$ then the elements of the sequence $\{x_n\}$ come frequently inside every neighbourhood $(x-\epsilon, x+\epsilon)$, $\epsilon > 0$; i.e. $\forall$ positive integer n there exists $m > n$ such that $x_m \in (x-\epsilon, x+\epsilon)$.

The following theorem, often called the Bolzano Weierstrass theorem, gives an existence criterion for cluster points.

Theorem 2.8 Every bounded sequence in $I\!R$ has a convergent subsequence.

Proof: (Sketch). Let $\{x_n\}$ be a bounded sequence such that the set $\{x_1, x_2, \cdots\} \subset [a,b]$. Divide this interval into two equal parts. Let I_1 be that interval which contains an infinite number of elements of $\{x_n\}$. Let x_{n_1} be one of the elements belonging to the interval I_1. Divide I_1 into two equal parts and let I_2 be that interval which contains an infinite number of elements. Choose a point x_{n_2} in I_2 such that $n_1 > n_2$. Keep dividing the intervals I_k, to generate I_k's and x_{n_k}'s. By nested interval theorem $\bigcap_{k=1}^{\infty} I_k = \{x\}$, for some $x \in [a,b]$. Then the subsequence $\{x_{n_k}\}$ converges to x. $\square$

Let S be an infinite set of numbers, then x is called a cluster point of S if for every $\epsilon > 0$, there exists an infinite number of elements $y \in S$ such that $|x - y| < \epsilon$.

Corollary 2.1 If S is a bounded and infinite subset of $I\!R$ then S has a cluster point.

If $\{x_n\}$ is a convergent sequence in $I\!R$ and if it converges to x, then we can roughly say that the size of the element x_n is "almost" x when n is large. Many times we deal with sequences which do not converge. Informally, the limit supremum of a sequence $\{x_n\}$ is a measure of how big x_n can be when n is large. Similarly, limit infimum is a measure of how small x_n can be when n is large. Before defining limit infimum and limit supremum formally, let us consider an example.

Consider the sequence,

$$\{x_n\} = 1,\ -1,\ 1,\ \frac{1}{2},\ -2,\ 1+\frac{1}{2},\ \frac{1}{3},\ -3,\ 1+\frac{1}{3},\ \ldots$$

This sequence is not convergent and it is not even bounded. We guess that the limit supremum of (x_n) is 1. However, 1 is not the supremum of the set $\{x_1, x_2, \ldots\}$, because there are elements which lie to the right of 1. But if we look slightly right side of 1, i.e., if we take $1+\epsilon$, $\epsilon > 0$, then ultimately all elements of the sequence lie to the left of $1+\epsilon$. So we guess that if x is the limit supremum of the sequence $\{x_n\}$, then x satisfies the following condition: "for every $\epsilon > 0$, $\exists\, N$ such that $x_n \leq x + \epsilon \;\forall\, n \geq N$." If an element x satisfies this condition then x need not be the limit supremum. For example, consider the above mentioned example, then any $x \geq 2$ satisfies this condition. But any element x which is greater than or equal to 2 is totally "away" from the sequence. So we require additional conditions to define the limit supremum.

Let us now state the formal definition of limit supremum.

Definition 2.3 *Let $\{x_n\}$ be a sequence of real numbers. Suppose there is a real number x satisfying the following conditions:*

1. *for every $\epsilon > 0$, there exists N such that $x_n \leq x + \epsilon$, $\forall\, n \geq N$*
2. *for given $\epsilon > 0$ and $m > 0$, there exists an integer $n > m$ such that $x_n \geq x - \epsilon$.*

It is clear that there can't be more than one x satisfying both conditions (i) and (ii). If an element x satisfies both (i) and (ii), then we say that x is the limit supremum of $\{x_n\}$ and write $x = \limsup\limits_{n\to\infty} x_n$.

If the sequence (x_n) is not bounded above, then we define $\limsup\limits_{n\to\infty} x_n = \infty$.

If the sequence is bounded above but not bounded below and if $\{x_n\}$ has no finite limit supremum, then we define $\limsup\limits_{n\to\infty} x_n = -\infty$.

Similarly we can define limit infimum of a sequence (x_n), which will be denoted by $\liminf\limits_{n\to\infty} x_n$. One can also verify that x is the limit infimum of x_n if and only if $x = -\limsup\limits_{n\to\infty}(-x_n)$.

We note that every sequence has a limit supremum and limit infimum in the extended real number system.

Examples

$$
\begin{array}{lll}
\text{(i)} \quad a_n = (-1)^n \left(1 + \frac{1}{n}\right) & \liminf\limits_{n\to\infty} a_n = -1 & \limsup\limits_{n\to\infty} a_n = 1 \\
\text{(ii)} \quad a_n = (-1)^n & \liminf\limits_{n\to\infty} a_n = -1 & \limsup\limits_{n\to\infty} a_n = 1 \\
\text{(iii)} \quad a_n = (-1)^n \cdot n & \liminf\limits_{n\to\infty} a_n = -\infty & \limsup\limits_{n\to\infty} a_n = \infty
\end{array}
$$

The proof of the following theorem is easy, hence it is left as an exercise.

Theorem 2.9

Let (x_n) be a sequence of real numbers. Then we have

1. $\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$.
2. $\{x_n\}$ converges to x if $\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$.
3. If $\{y_n\}$ is sequence, such that $x_n \leq y_n$. Then $\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} y_n$ and $\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n$.