Differentiable Functions

In this chapter we deal with derivatives, an important concept of differential calculus. The reader must be familiar with this from elementary calculus. For example, the geometric problem of finding the tangent line to a curve at a given point leads to the notion of derivative.

Let I be an interval which is not a singleton and let f be a function defined on I. A function f is said to be differentiable at $x \in I$ if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. It is understood that the limit is taken for $x + h \in I$, thus if x is a left end point of I then we only consider $h > 0$. If the above limit exists, it is called the derivative of f at x and is denoted by $f'(x)$. If f is differentiable at each $x \in I$, then f' is a function on I.

It is clear that if f is differentiable at $c \in I$, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

Now we prove that differentiability implies continuity.

Theorem 4.1 Let f be defined on an interval I. If f is differentiable at a point $c \in I$, then f is continuous at c.

Proof: As $x \to c$, $x \neq c$, we have

$$f(x) - f(c) = \frac{f(x) - f(c)}{x - c}\ (x - c) \ \to\ f'(x) \cdot 0 = 0.$$

□

The proof of the following theorem is similar to the proof of the previous theorem; hence it is left as an exercise.

Theorem 4.2 If f is defined on an interval I and differentiable at a point $c \in I$, then there is a function f^* which is continuous at c and which satisfies the equation

$$f(x) - f(c) = (x - c)\ f^*(x) \tag{1}$$

for all $x \in I$, with $f^*(c) = f'(c)$.

Conversely, if there is a function f^*, continuous at c, which satisfies Equation 1, then f is differentiable at c and $f'(c) = f^*(c)$.

As an immediate consequence of the previous theorem, we have the following remark. Since f^* is continuous at c, we have $f^*(x) \to f^*(c) = f'(c)$ as $x \to c$. Replacing $f^*(x)$ by $f'(c)$ in Equation 1, we obtain the "equation"

$$f(x) \approx f(c) + f'(c)\ (x - c),$$

which should be approximately correct when $x - c$ is small. In other words, if f is differentiable at c, then f is approximately a linear function near c. This is illustrated in the following figure.

The proof of the following two theorems are left as exercises.

Theorem 4.3 Suppose f and g are defined on I and are differentiable at a point $c \in I$. Then

1. $(f+g)'(c) = f'(c) + g'(c)$
2. $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$
3. if $g(c) \neq 0$, then $(\frac{f}{g})'(c) = \frac{g(c)\ f'(c) - g'(c)f(c)}{g^2(c)}$.

The following theorem is known as *chain rule.*

Theorem 4.4 Let f, g be defined on intervals I, J respectively, and suppose $f(I) \subset J$. Let f be differentiable at x and g differentiable at $f(x)$. Then $g_0 f$ is differentiable at x and

$$(gof)'(x) = g'((f(x))\ f'(x).$$

4.1 Rolle's Theorem and Mean-Value Theorem

The reader must be familiar with the classical maxima and minima problems from calculus. For example, the graph of a differentiable function has a horizontal tangent at a maximum or minimum point. This is not quite accurate but this will be clear now.

Definition 4.1 Let $f : I \to I\!R$, I an interval. A point $x_0 \in I$ is a local maximum of f if there is a $\delta > 0$ such that if $x \in I \cap (x_0 - \delta, x_0 + \delta)$ then

$$f(x) \leq f(x_0).$$

Similarly, we can define local minimum.

Theorem 4.5 Suppose $f : [a, b] \to I\!R$ and suppose f has either a local maximum or a local minimum at $x_0 \in (a, b)$. If f is differentiable at x_0 then $f'(x_0) = 0$.

Proof: (Sketch): Suppose f has a local maximum at $x_0 \in (a, b)$. For small (enough) h, $f(x_0 + h) \leq f(x_0)$.
If $h > 0$ then

$$\frac{f(x_0 + h) - f(x_0)}{h} \leq 0.$$

Similarly, if $h < 0$, then

$$\frac{f(x_0 + h) - f(x_0)}{h} \geq 0.$$

By elementary properties of the limit, it follows that $f'(x_0) = 0$. $\Box$

We remark that the previous theorem is not valid if x_0 is a or b. For example, if we consider the function $f : [0, 1] \to I\!R$ such that $f(x) = x$, then f has maximum at 1 but $f'(x) = 1\ \forall\ x \in [0, 1]$.

The following theorem is known as *Rolle's theorem* which is an application of the previous theorem.

Theorem 4.6 Let f be continuous on $[a, b]$, $a < b$, and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists c such that $c \in [a, b]$ and $f'(c) = 0$.

Proof: If f is constant on $[a, b]$, $f'(c) = 0\ \forall\ c \in [a, b]$. Suppose there exists $x \in (a, b)$ such that $f(x) > f(a)$. (A similar argument can be given if $f(x) < f(a)$). Then there exists $c \in (a, b)$ such that $f(c)$ is a maximum. Hence by previous theorem,we have $f'(c) = 0$. □

A geometric interpretation of the above theorem can be given as follows. If the values of a differentiable function f at the end points a and b are equal then somewhere between a and b there is a horizontal tangent. It is natural to ask the following question. If the value of f at the end points a and b are not the same, is it true that there is some $c \in [a, b]$ such that the tangent line at c is parallel to the line connecting the endpoints of the curve? The answer is yes and this is essentially the Mean-Value Theorem.

Theorem 4.7 (Mean Value Theorem)
Let f be continuous on $[a, b]$, and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.

Proof: Let

$$g(x) = f(x) - \frac{f(b) - f(a)}{b - a}\,(x - a).$$

Then $g(a) = g(b) = f(a)$. The result follows by applying Rolles Theorem to g. □

The mean value theorem has some important applications relating the behaviour of f and f'.

A function f on an interval is called increasing if $f(x) \leq f(y)$ whenever $x \leq y$. It is called strictly increasing if $f(x) < f(y)$ whenever $x < y$. Similarly, we can define decreasing and strictly decreasing functions.

Theorem 4.8 Let f be continuous on $[a, b]$ and differentiable on (a, b). Suppose $f'(x) > 0$ for $a < x < b$. Then f is strictly increasing on $[a, b]$.

Proof Let $a \leq x < y \leq b$. By mean value theorem

$$f(y) - f(x) = f'(c)(y - x)$$

for some $c \in (x, y)$. Since $y - x > 0$, we conclude that f is strictly increasing. □

Similar conclusions can be drawn if $f'(x) < 0, f'(x) \geq 0, f'(x) \leq 0$.

The proof of the following theorem is similar to the proof of the previous theorem, hence it is left as an exercise.

Theorem 4.9 Suppose f is continuous on $[a, b]$ and differentiable on (a, b). Then we have the following:

1. If $f'(x) \neq 0\ \forall\ x \in (a, b)$, then f is one-one.
2. If $f'(x) = 0\ \forall\ x \in (a, b)$, then f is constant.

Let f be differentiable on $[a, b]$. If $f'(x) \neq 0\ \forall\ x \in (a, b)$, then by the previous theorem f is one-one. Hence by intermediate value theorem, either f is increasing or decreasing. If f is increasing, then, for all $x, y \in [a, b]$ with $x \neq y$ we have

$$\frac{f(x) - f(y)}{x - y} > 0.$$

This implies that $f'(x) \geq 0 \, \forall \, x \in [a, b]$. Similarly, if f is decreasing then $f'(x) \leq 0 \, \forall \, x \in [a, b]$.

From the above discussion we can conclude the following: If f is differentiable on $[a, b]$ and there exists $x, y \in [a, b]$ such that $f'(x) > 0$ and $f'(y) < 0$, then there exists $c \in (x, y)$ such that $f'(c) = 0$. So it seems that f' has the intermediate value property.

Theorem 4.10 Let f be differentiable on $[a, b]$. Let λ be a real number such that $f'(a) < \lambda < f'(b)$ or $f'(b) < \lambda < f'(a)$. Then there is $c \in (a, b)$ such that $f'(c) = \lambda$.

Proof: If we define $g(x) = f(x) - \lambda \, x \, \forall \, x \in [a, b]$ then g is differentiable on $[a, b]$ and $g'(x) = f'(x) - \lambda$. If $f'(a) < \lambda < f'(b)$, then $g'(a) < 0$ and $g'(b) > 0$; if $f'(b) < \lambda < f'(a)$, then $g'(a) > 0$ and $g'(b) < 0$. Hence g' has opposite signs at a and b. Hence there is $c \in (a, b)$ such that $0 = g'(c) = f'(c) - \lambda$ so that $f'(c) = \lambda$.

The theorem reveals that not every function is a derivative.

Inverse-function Theorem

We have already observed that (see Problems of the previous chapter) if $f : [a, b] \to I\!R$ is continuous and one-one, then f^{-1} is also continuous. It is natural to ask the following: If a function is one-one and differentiable then is f^{-1} also differentiable?

We will now answer this question.

Suppose f is one-one, differentiable and f^{-1} is also differentiable. Then by chain rule

$$(f \mathrm{o}^{-1} f)'(x) = (f^{-1})' \; (f(x)) f'(x)$$

for all $x \in [a, b]$. This implies

$$(f^{-1})' \; (f(x)) f'(x) = 1$$

for all $x \in [a, b]$. Hence

$$(f^{-1})' (f(x)) = \frac{1}{f'(x)}.$$

These observations motivate the following result.

Theorem 4.11 Suppose $f : [a, b] \to I\!R$ is differentiable. Suppose $f'(x) \neq 0 \; \forall \; x \in [a, b]$. Then f is one-one, f^{-1} is continuous and differentiable on $f[(a, b)]$, and

$$(f^{-1})' \; (f(x)) = \frac{1}{f'(x)}$$

for all $x \in [a, b]$.

Proof: We have already seen that if $f'(x) \neq 0 \; \forall \; x \in [a, b]$ then f is one-one. Suppose $f([a, b]) = [c, d]$. We will show that $f^{-1} : [c, d] \to [a, b]$ is differentiable. Let $c \leq y_0 \leq d$, $y_0 = f(x_0)$ and $y = f(x)$ for some $x \in [a, b]$. Then

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}}.$$

Now let $y \to y_0$ and use the continuity of f^{-1} to get the result.