CHAPTER 4

Continuous Function

Intuitively, we think of a function $f : \mathbb{R} \to \mathbb{R}$ as continuous if it has a continuous curve. The term *continuous curve* means, intuitively, that the graph of f can be drawn without *jumps*, i.e., the graph can be drawn with a *continuous* motion of the pencil without leaving the paper.

Suppose a function $f : \mathbb{R} \to \mathbb{R}$ has a discontinuous graph as shown in the following figure.

The graph is discontinuous at the point $(x_0, f(x_0))$, i.e., the function f is discontinuous at x_0. Hence whenever x is close to x_0 from the right, $f(x)$ does not get close to $f(x_0)$. (The idea of getting close has already been discussed while dealing with convergent sequences). As shown in the figure, we can choose a neighbourhood $(f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$, $\epsilon_0 > 0$, at

$f(x_0)$ such that if we take any neighbourhood $(x_0 - \delta, x_0 + \delta)$, $\delta > 0$, then the image of the interval $(x_0 - \delta, x_0 + \delta)$ does not lie inside $(f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$. In formal terms, $\exists\ \epsilon > 0$ such that $\forall\ \delta > 0$, $|x - x_0| < \delta \not\Rightarrow |f(x) - f(x_0) < \epsilon$. Hence if a function f is not continuous at x_0, we have the above condition.

We will now give the formal definition of continuity of a function at a point (in the "ϵ-δ language").

Definition A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at a point $x_0 \in \mathbb{R}$ if for every $\epsilon > 0$, there is $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_o| < \delta$.

Roughly, f is continuous at x_0 if whenever x approaches x_0, $f(x)$ approaches $f(x_0)$. In some cases when f is not continuous at x_0, there may be a number A such that whenever x approaches x_0, $f(x)$ approaches A. In this case we call such a number A the limit point of f at x_0. Formally, we have:

A number A is called the limit of a function f at a point x_0 if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - A| < \epsilon$ whenever $0 < |x - x_0| < \delta$. If such a number A exists then it is unique.

In this case we write $\lim\limits_{x \to x_0} f(x) = A$. It is clear that $f(x_0)$ is the limit of f at x_0 if f is continuous at x_0.

The reader is advised to see the strong analogy between the definition of limit point and the definition of convergence of sequence.

Let us now characterize the continuity of a function at a point in terms of sequences.

Theorem 3.1 A real valued function f is continuous at $x_0 \in \mathbb{R}$ if and only if whenever a sequence of real numbers $\{x_n\}$ converges to x_0, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$.

Proof: Suppose f is continuous at x_0 and $x_n \to x_0$. Let us show that $f(x_n) \to f(x_0)$. Let $\epsilon > 0$ be given. We must find N such that $|f(x_n) - f(x_0)| < \epsilon\ \forall\ n \geq N$. Since f is continuous at x_0, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. Since $x_n \to x_0$, there exists N such that $|x_n - x_0| < \delta\ \forall\ n \geq N$. This N serves our purpose.

To prove the converse, let us assume the contrary that f is not continuous at x_0. Then for some $\epsilon > 0$ and for each n, there is an element x_n such that $|x_n - x_0| < \frac{1}{n}$ but $|f(x_n) - f(x_0)| \geq \epsilon$. This contradicts the fact that $x_n \to x_0$ implies $f(x_n) \to f(x_0)$.

To define the continuity of a function f at a point x_0, the function f has to be defined at x_0. But even if the function is not defined at x_0, one can define the limit of a function at x_0.

Let S be a subset of $\mathbb{R}$. An element $x_0 \in S$ is said to be a limit point of S if for every $\epsilon > 0$, there exists $x \in S$ such that $|x - x_0| < \epsilon$.

Let f be a function defined on a subset S of $\mathbb{R}$ and let x_0 be a limit point of S, we say that L is the limit point of f at x_0, if for every $\epsilon > 0$, $\exists\ \delta > 0$ such that whenever $x \in S$ with $|x - x_0| < \delta$ we have $|f(x) - L| < \epsilon$.

Similarly, we can define a continuous function on a subset of $\mathbb{R}$. Let S be a subset of $\mathbb{R}$ and $x_0 \in S$, we say that f is continuous at x_0, if for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x \in S$ with $|x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \epsilon$. Moreover, if f is continuous at each $x \in S$, then we say that f is continuous on S.

Examples

1. Let us consider the function $f(x) = \frac{x^2-4}{x-2}$. This function f is defined for all $x \neq 2$. Let us find its limit as $x \to 2$. Note that f is not defined at $x = 2$. But 2 is a limit point of the set $\{x : x \neq 2\}$.

   For any $x \neq 2$, we have $\frac{x^2-4}{x-2} = x + 2$. Since the definition of the limit as $x \to 2$ does not involve the value of the function f at $x = 2$, we have

   $$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2}(x + 2) = 4.$$

   It should be understood that $f(x) = \frac{x^2-4}{x-2}$ and $g(x) = x+2$ are different functions. But for evaluating the limit of f as $x \to 2$, we are not interested in whether f is defined at $x = 2$ or not.

   We also note that this function f is continuous wherever it is defined.

2. It is clear that $\lim\limits_{x \to 3} x^2 = 9$. Let us prove this by using the "$\epsilon - \delta$ language". Given $\epsilon > 0$ we must find $\delta > 0$ such that $|x^2 - 9| < \epsilon$ whenever $|x - 3| < \delta$. We note that $|x^2 - 9| = |x - 3|\ |x + 3|$ and that we may choose $\delta < 1$. Let us see how big $|x + 3|$ is. If $|x - 3| < \delta < 1$ then $|x + 3| < 7$. Hence $|x^2 - 9| < \delta \cdot 7$ if $|x - 3| < \delta < 1$. So let $\delta = \min(1, \frac{\epsilon}{7})$.

3. The function $\sin(1/x)$ is defined for all $x \neq 0$. Moreover, 0 is the limit point of $S = \{x : x \neq 0\}$. The graph of the function is given below.

   This function has no limit as $x \to 0$ because if we take $x_n = \{2/\pi(2n + 1)\}$ for $n = 1, 2, \ldots$, then $x_n \to 0$ but $f(x_n) = (-1)^n$ which does not tend to any limit as $n \to \infty$.

3.1 Limits at Infinity

We say that the set S contains arbitrarily large numbers if for any number y there exist $x \in S$ such that $x \geq y$. Let f be defined on S. We say that $\lim_{x \to \infty} f(x)$ exists if $\exists$ a number A such that for every $\epsilon > 0$, there exist $y > 0$ such that whenever $x \in S$, $x \geq y$, we have $|f(x) - A| < \epsilon$.

Let S be a set of numbers and let x_0 be a limit point of S. We say that $\lim_{x \to x_0} f(x) = \infty$ for a function f defined on S if for every M, $\exists\, \delta > 0$ such that wherever $x \in S$ and $|x - x_0| < \delta$ implies $f(x) > M$.

The proof of the following theorem is left as an exercise.

Theorem 3.2 Let x_0 be a limit point of the set S. Let f and g be functions defined on S. Suppose $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = M$. Then

1. $\lim_{x \to x_0} (f + g)(x) = L + M$
2. $\lim_{x \to x_0} (fg)(x) = LM$
3. If $g(x) \neq 0\ \forall\ x \in S$ and if $M \neq 0$ then $\lim_{x \to x_0} (\frac{f}{g})(x) = \frac{L}{M}$.

Problems

1. Prove that $\lim_{x \to 0} x \ \sin\ 1/x = 0$.
2. Define $f : I\!R \to I\!R$ as

$$\begin{aligned} f(x) &= x \quad \text{if} \quad x \quad \text{is rational} \\ &= 0 \quad \text{if} \quad x \quad \text{is irrational.} \end{aligned}$$

   Then $\lim_{x \to c} f(x)$ exists if $c = 0$.

3. Define $f : (0, 1) \to I\!R$ by

$$\begin{aligned} f(x) &= \frac{1}{n} \quad \text{if} \quad x = \frac{m}{n} \text{ is rational and the fraction is in reduced form} \\ &= 0 \quad \text{if} \quad x \quad \text{is irrational.} \end{aligned}$$

   Show that f is continuous at c if c is irrational.

4. If $f : I\!R \to I\!R$ is continuous and $f(x) = 0$ for all rational x, show that $f(x) = 0\ \forall\ x \in I\!R$.
5. Suppose $f : I\!R \to I\!R$ such that $f(x + y) = f(x) + f(y)$ foy every $x, y \in I\!R$. If f is continous then show that $f(x) = cx$ for some $c \in I\!R$.

3.2 Properties of Continuous Functions on a Closed Interval

For $a, b \in I\!R$, let

$$[a, b] = \{x \in I\!R : a \leq x \leq b\}.$$

We call $[a, b]$ a closed interval of $I\!R$.

We will now see some properties of continuous functions on a closed interval.

Theorem 3.3 If a function f is continuous on a closed interval $[a, b]$ then it is bounded on $[a, b]$.

Proof: Suppose that f is not bounded on $[a, b]$. Then for each natural number n there is a point $x_n \in [a, b]$ such that $|f(x_n)| > n$, for all $n = 1, 2, \ldots$. Since (x_n) is a bounded sequence, by Bolzano-Weierstrass theorem it has a convergent subsequence say $x_{n_k} \to x_0 \in [a, b]$. By the continuity of f, we have $f(x_{n_k}) \to f(x_0)$. This contradicts the assumption that $|f(x_n)| > n \; \forall \; n$. Hence$f$ is bounded on $[a, b]$. $\square$

We remark that if a function is continuous on an open interval (a, b) or on a semi-open interval of the type $(a, b]$ or $[a, b)$, then it is not necessary that the function has to be bounded. For example, consider the continuous function $\frac{1}{x}$ on $(0, 1]$.

Let f be defined on a subset S of $I\!R$. An element $x_0 \in S$ is called a maximum for f on S if $f(x_0) \geq f(x) \; \forall \; x \in S$. Similarly, x_0 is called a minimum for f on S if $f(x_0) \leq f(x) \; \forall \; x \in S$.

The following theorem provides an existence criterion for the maximum and minimum.

Theorem 3.4 Let f be continuous on $[a, b]$. Then there exist $x_0, y_0 \in [a, b]$ such that x_0 is a maximum for f on $[a, b]$ and y_0 is a minimum for f on $[a, b]$.

Proof: By the previous theorem f is bounded on $[a, b]$. Let M be the least upper bound of $f([a, b])$ $(:= \{f(x) : x \in [a, b]\})$. Then there exist a sequence $\{f(x_n)\}$ in $f([a, b])$ such that $f(x_n) \to M$. Since $\{x_n\}$ is a bounded sequence in $[a, b]$, it has a convergent subsequence, say $x_{n_k} \to x_0 \in [a, b]$. By the continuity of f we have $f(x_0) = M$. Hence x_0 is a maximum for f on $[a, b]$.

The proof for the minimum is similar. $\square$

Consider a function $f : [-1, 1] \to I\!R$ such that f is continuous and satisfies $f(-1) < 0$ and $f(1) > 0$. Intuitively, we feel that the graph of f should cross the x-axis between -1 and 1; ie., there is some $x_0 \in [-1, 1]$ such that $f(x_0) = 0$. This motivates us to state the following theorem.

Theorem 3.5 Intermediate Value Theorem: Let f be continuous on $[a, b]$, and let $f(a) < s < f(b)$. Then there exist x such that $a < x < b$ and $f(x) = s$.

Proof: Let $S = \{x \in [a, b] : f(x) \leq s\}$. Since $a \in S$, we have $S \neq \emptyset$. Let c be the least upper bound of S. We claim that $f(c) = s$. Since c is the least upper bound of S, there exists a sequence $\{x_n\}$ from S such that $x_n \to c$. Note that $c \in [a, b]$. Hence $f(x_n) \to f(c)$. Since $f(x_n) \leq s \; \forall \; n$, we have $f(c) \leq s$. If $x \in [a, b]$ and $x > c$, then $f(x) > s$, since c is an upper bound for S. Let $T = \{x \in [a, b] : x > c\}$. Then $b \in T$ and so $T \neq \emptyset$. Since c is a limit point of the set T, there exist a sequence (y_n) from T such that $y_n \to c$. Hence $f(y_n) \to f(c)$. Since $f(y_n) > s$, we have $f(c) \geq s$. It follows that $f(c) = s$.

Problems

1. Let $f : [a, b] \to I\!R$ be continuous and one-one. Show that f^{-1} is also continuous.(Note that here f^{-1} is defined range of f.)

2. Let f be continuous on $[a, b]$. Show that the image of f, $f([a, b])$ is a closed interval. Is the image a subinterval of $[f(a), f(b)]$?

3. Show that the function $f(x) = \frac{x^2}{x^2+1}$, $x \in I\!R$ is bounded but has no maximum on $I\!R$.

4. If f is a polynomial of odd degee then show that f has a real root.

5. Let f be a continuous function on $I\!R$ such that$f(f(x)) = x \; \forall \; x$. Show that there exists a point c such that $f(c) = c$.

6. If f is a continuous function from $[a, b]$ into $[a, b]$ then there is at least one $x \in [a, b]$ for which $f(x) = x$.

7. Let $S \subset I\!R$ and $f : S \to I\!R$. We say that f is uniformly continuous on S if for given $\epsilon > 0$, there exists $\delta > 0$ such that whenever $x, y \in S$ and $|x - y| < \delta$, we have $|f(x) - f(y)| < \epsilon$.