Fundamental Theorems of Calculus

Integration and differentiation are, in a sense, inverse operations.

Theorem (First Fundamental Theorem of Calculus.) : Let $f \in \mathcal{R}$ on $[a, b]$. For $a \le x \le b$, let $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a, b]$ and if f is continuous at x_0 then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof Suppose $M = \sup\{|f(x)| : x \in [a, b]\}$. Let $a \le x < y \le b$. Then

$$|F(y) - F(x)| = |\int_x^y f(t)dt| \le M(y - x).$$

Thus $|F(x) - F(y)| \le M|x - y|$ for $x, y \in [a, b]$. Hence F is continuous, infact uniformly continuous.

Now suppose f is continuous at x_0. Given $\epsilon > 0$ choose $\delta > 0$ such that

$$|t - x_0| < \delta \ \Rightarrow |f(t) - f(x_0)| < \epsilon.$$

Let x be such that $0 \le |x - x_0| < \delta$. Then

$$\left|\frac{F(x) - F(x_0)}{x - x_0} - f(x_0)\right| = \left|\frac{1}{x - x_0}\int_{x_0}^x [f(t) - f(x_0)]dt\right| < \epsilon.$$

This implies that $F'(x_0) = f(x_0)$. □

In the previous theorem, in a sense, we obtain f by differentiating integral of f when f is continuous on $[a, b]$. A function F such that $F'(x) = f(x)\ \forall\ x \in [a, b]$ is called an antiderivative of f on $[a, b]$. The existence of the antiderivative for a continuous function on $[a, b]$ follows from the first F.T.C.

If an integrable function f has the antiderivative (and if we can find it), then calculating its integral is very simple. The second F.T.C. explains this.

Theorem (Second Fundamental Theorem of Calculus): Let $f \in \mathcal{R}$ on $[a, b]$. If there is a differentiable function F on $[a, b]$ such that $F' = f$ then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof Let $\epsilon > 0$. Choose a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$. Now there exists a $c_i \in [x_{i-1}, x_i]$ such that $F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i$. Hence

$$\sum f(c_i)\Delta x_i = F(b) - F(a).$$

But we know that

$$L(P, f) \le \int f dx \le U(P, f)$$

and

$$L(P, f) \le \sum f(c_i)\Delta x_i \le U(P, f).$$

Therefore

$$|F(b) - F(a) - \int_a^b f dx| < \epsilon.$$

This completes the proof. □

It is clear from the second F.T.C. that why the indefinite integral of F' is defined to be F.