Handout 5, Functions of several variables.

For $X = (x_1, x_2, ..., x_m)$, $Y = (y_1, y_2, ..., y_m) \in \mathbb{R}^m$, $\| X \| = ({x_1}^2 + {x_2}^2 + ... + {x_m}^2)^{1/2}$ is the distance between 0 and X, and $\| X - Y \|$ is the distance between X and Y.

We say that a sequence $X_n = (x_{1,n}, x_{2,n}, ..., x_{m,n})$ converges to $X_0 = (x_{1,0}, x_{2,0}, ..., x_{m,0})$ if the elements of the sequence $\{X_n\}$ come eventually inside every neighbourhood of X_0, namely the set

$$\{Y \in \mathbb{R}^m : \| X_0 - Y \| < \epsilon\}, \quad \epsilon > 0.$$

More formally, $X_n \to X_0$ if for every $\epsilon > 0$, there exists N such that $\| X_n - X_0 \| < \epsilon$ for every $n \geq N$. In case $m = 1$, we use $| \ |$ in place of $\| \ \|$, hence it is clear how we generalized the concept of convergence of a sequence in $\mathbb{R}^1$ to $\mathbb{R}^m$. Similarly we say that $f : \mathbb{R}^m \to \mathbb{R}$ is continuous at $X_0 \in \mathbb{R}$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that $| f(X) - f(X_0) | < \epsilon$ whenever $\| X - X_0 \| < \delta$. It is now clear how to define limit of a function $f : \mathbb{R}^m \to \mathbb{R}$ at some point $X_0 \in \mathbb{R}^m$.

Note that $X_n = (x_{1,n}, x_{2,n}, ..., x_{m,n}) \to X_0 = (x_{1,0}, x_{2,0}, ..., x_{m,0})$ in $\mathbb{R}^m$ if and only if the coordinates $x_{i,n} \to x_{i,0}$ for every $i = 1, 2, ..., m$ in $\mathbb{R}$.

The proof of the following characterization runs on the same line as the proof of corresponding result of single variable by replacing $| \ |$ by $\| \ \|$ in the domain of f.

Theorem 5.1: A function $f : \mathbb{R}^m \to \mathbb{R}$ is continuous at $X \in \mathbb{R}^m$ if and only if whenever a sequence $\{X_n\}$ in $\mathbb{R}^m$ converges to X then the sequence $f(X_n) \to f(X)$.

For simplicity, now onwards we assume $f : \mathbb{R}^3 \to \mathbb{R}$. General case is entirely analogous.

Partial derivatives :

The partial derivative of f w.r.to the first variable at $X_0 = (x_0, y_0, z_0)$ is defined by

$$\frac{\partial f}{\partial x} |_{X_0} = \lim_{h \to 0} \frac{f(x_0 + h, y_0, z_0) - f(x_0, y_0, z_0)}{h}$$

provided the limit exists. Similarly we define $\frac{\partial f}{\partial y} |_{X_0}$ and $\frac{\partial f}{\partial z} |_{X_0}$.

Example 1 One can verify that the function

$$f(x, y) = \frac{2xy}{x^2 + y^2} \ at \ (x, y \neq (0, 0)$$

$$= 0 \ at \ (0, 0)$$

is not continuous at $(0, 0)$, however the partial derivatives exist at $(0, 0)$.

Differentiability :

When $f : \mathbb{R} \to \mathbb{R}, x \in \mathbb{R}$ we define

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \qquad (*)$$

provided the limit exists. In case $f : \mathbb{R}^3 \to \mathbb{R}$ and $X = (x_1, x_2, x_3) \in \mathbb{R}^3$ the above definition of the differentiability of functions of one variable (*) cannot be generalized as we cannot divide by an element of $\mathbb{R}^3$. What we do is to rearrange the above definition (*) to a form which can be generalized.

Let $f : \mathbb{R} \to \mathbb{R}$. Then f is differentiable at x if and only if there exists $\alpha \in \mathbb{R}$ such that

$$\frac{\mid f(x+h) - f(x) - \alpha \cdot h \mid}{\mid h \mid} \to 0 \quad as \quad h \to 0.$$

When f is differentiable at x, α has to be $f^{'}(x)$. We generalize this definition to the functions of several variables.

Definition : Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $X = (x_1, x_2, x_3)$. We say that f is *differentiable* at X if there exists $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ such that the error function

$$\varepsilon(H) = \frac{f(X+H) - f(X) - \alpha \cdot H}{\parallel H \parallel}$$

tends to 0 as $H \to 0$.

In the above definition $\alpha \cdot H$ is the scalar product. Note that the derivative $f^{'}(X) = (\alpha_1, \alpha_2, \alpha_3)$.

Theorem 5.2 : Let $f : \mathbb{R}^3 \to \mathbb{R}$, $X \in \mathbb{R}^n$. If f is differentiable at X then f is continuous at X.

Proof : Suppose f is differentiable at X. Then there exists $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$ such that

$$\mid f(X+H) - f(X) - \alpha \cdot H \mid = \parallel H \parallel \varepsilon(H) \text{ and } \varepsilon(H) \to 0 \text{ as } H \to 0.$$

Hence

$$\mid f(X+H) - f(X) \mid \leq \parallel H \parallel (\sum_{i=1}^{3} \mid \alpha_i \mid) + \parallel H \parallel \varepsilon(H)$$

and $\varepsilon(H) \to 0$ as $H \to 0$. Therefore $f(X+H) \to f(X)$ as $H \to 0$. This proves that f is continuous at X.

How do we verify that a given function is differentiable at a point in $\mathbb{R}^3$? The following result helps us to answer this question.

Theorem 5.3: Suppose f is differentiable at X. Then the derivative

$$f^{'}(X) = (\alpha_1, \alpha_2, \alpha_3) = (\frac{\partial f}{\partial x}\mid_X, \frac{\partial f}{\partial y}\mid_X, \frac{\partial f}{\partial z}\mid_X).$$

Proof : Suppose f is differentiable at X and $f^{'}(X) = (\alpha_1, \alpha_2, \alpha_3)$. Then by taking $H = (t, 0, 0)$, we have

$$\varepsilon(H) = \frac{f(X+H) - f(X) - \alpha_1 t}{t} \to 0 \quad as \quad t \to 0$$

This implies that $\alpha_1 = \frac{\partial f}{\partial x}\mid_X$.

Example 2 : Let

$$f(x,y) = xy\frac{x^2 - y^2}{x^2 + y^2} \quad at \quad (x,y) \neq (0,0)$$

$$= 0 \qquad at(0,0)$$

To verify that f is differentiable at $(0,0)$, let us choose $\alpha = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) \mid_{(0,0)}$ and verify that $\epsilon(H) \to 0$ as $H \to 0$. In this case $\alpha = (0,0)$ and

$$\varepsilon(H) = \frac{f(0+H) - f(0) - (0,0) \cdot H}{\parallel H \parallel} \to 0 \quad as \quad H \to 0.$$

Hence f is differentiable at $(0,0)$.

Example 1 illustrates that partial derivatives of a function may exist at a point but the function need not be differentiable at that point.