Handout 1, The Real Number System

In calculus, we deal with functions of a real variable. In this note we will give some idea about the real number system and its properties.

We start with the set of integers. We know that given any two integers, these can be added, one can be subtracted from the other and they can be multiplied. The result of each of these operations is again an integer. Further, if p and q are integers and $p - q > 0$, then we say that $p > q$ (this defines an order relation '$<$' on the set of integers.)

Next we consider numbers of the form m/n, where m and n are integers and $n \neq 0$, called rational numbers. (We may assume that m and n have no common factor.) The operations of addition (and subtraction) multiplication and '$<$' extend to this set in a natural way. We shall see, from a very simple situation that numbers other than rational numbers are needed. Consider a square whose side has unit length. Then by Pythagora's Theorem, the length l of the diagonal must satisfy $l^2 = 2$ (we write $l = \sqrt{2}$). What is l? Suppose $l = m/n$, where m and n are integers, which are not both even. Then $l^2n^2 = 2n^2 = m^2$. Thus m^2 is even, since the square of an odd integer is odd, we conclude that m is even, so that n^2 is even, and hence, n is divisible by 2. This contradicts our assumption.

The above discussion shows that l is not a rational number, and so we need numbers such as $\sqrt{2}$- these will be called irrational numbers. The problem now is to define these numbers

We will not present the definition of real numbers here, *since it is bit involved.* Instead we will give a rough idea about real numbers.

On a straight line, if we mark off segments ..., $[-1, 0], [0, 1], [1, 2], \ldots$ then all the rational numbers can be represented by points on this straight line. The set of points representing rational numbers seems to fill up this line (rational number $\frac{r+s}{2}$ lies in between the rational numbers r and s). But we have seen above that the rationals do not cover the entire straight line. Intuitively we feel that there should be a larger set of numbers, say $\mathbb{R}$ such that there is a correspondence between $\mathbb{R}$ and the points of this straight line. Indeed, one can construct such a set of numbers from the rational number system $\mathbb{Q}$, called set of real numbers, which contains the set of rationals and also numbers such as $\sqrt{2}, \sqrt{3}, \sqrt{5}$ and more. Moreover, on this set we can define operations of addition and multiplication, and an order in such a way that when these operations are restricted to the set of rationals, they coincide with the usual operations and the usual order. The set $\mathbb{R}$ with these operations is called the real number system.

The important property of $\mathbb{R}$, which is missing in $\mathbb{Q}$ is the following.

Completeness property of real number system:

A subset A of $\mathbb{R}$ is said to be *bounded above* if there is an element $x_0 \in \mathbb{R}$ such that $x \leq x_0$ for all $x \in A$. Such an element x_0 is called an *upper bound* of A . Similarly A is said to be *bounded below* if there exists $y_0 \in \mathbb{R}$ such that $y_0 \leq x$ for all $x \in \mathbb{R}$.

An upper bound x_0 of A is said to be a *least upper bound* (l.u.b.) or supremum (sup) of A if whenever z is an upper bound of A, $x_0 \leq z$. A *greatest lower bound* (g.l.b.) or infimum (inf) is defined similarly.

Remark. The least upper bound or the greatest lower bound may not belong to the set A.

Real number system has the property that every non-empty subset of $\mathbb{R}$ which is bounded above has a least upper bound. This property is called *least upper bound property.* The greatest lower bound property is defined similarly. *Completeness property* we mean either l.u.b. property or g.l.b. property.

Let $A = \{r \in \mathbb{Q} \,:\, r > 0, r^2 < 2\}$; this is a non-empty and bounded subset of $\mathbb{Q}$. The set A does not have l.u.b. in $\mathbb{Q}$. This shows that $\mathbb{Q}$ does not have the least upper bound property.

We will now show that the following properties which will be used later on are important consequences of completeness property of $\mathbb{R}$.

Proposition 1.1 (Archimedean property): *If $x, y \in \mathbb{R}$ and $x > 0$, then there is a positive integer n such that $nx > y$.*

Proof. Suppose that $nx \leq y$ for every positive integer n. Then y is an upper bound of the set $A = \{nx \,:\, n \in \mathbb{N}\}$. By the least upper bound property, let α be a l.u.b. of A. Then $(n+1)x \leq \alpha$ for all n and so $nx \leq \alpha - x < \alpha$ for all n i.e. $\alpha - x$ is also an upper bound which is smaller than α. This is a contradiction. $\square$

The Archimedean property leads to the *"density of rationals in $\mathbb{R}$"* and *"density of irrationals in $\mathbb{R}$".*

Proposition 1.2: *Between any two distinct real numbers there is a rational number.*

Proof. Suppose $x, y \in I\!R$, $y - x > 0$. We have to find two integers m and n, $n \neq 0$ such that

$$\begin{array}{rrcl} & x < & \dfrac{m}{n} & < y \\ \text{i.e.} & x < & \dfrac{m}{n} & < x + (y - x). \end{array}$$

Now by the Archimedean property there exists a positive integer n such that $n(y-x) > 1$. Then we can find an integer m lying between nx and $ny = nx + n(y-x)$. This will do! $\square$

Proposition 1.3 : *Between any two distinct real numbers there is an irrational number.*

Proof: Suppose $x, y \geq 0$, $y - x > 0$. Then $\frac{1}{\sqrt{2}}x < \frac{1}{\sqrt{2}}y$. By Proposition 1.2, there exists a rational number r such that $x < r\sqrt{2} < y$. $\square$