Handout 3, Riemann Integral

Integration is concerned with the problem of finding the area of a region under a curve.

Let us start with a simple problem : *Find the area A of the region enclosed by a circle of radius r.* For an arbitrary n, consider the n equal inscribed and superscibed triangles as shown in Figure 1.

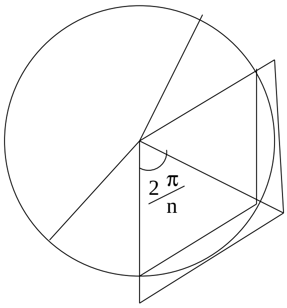

Figure 1

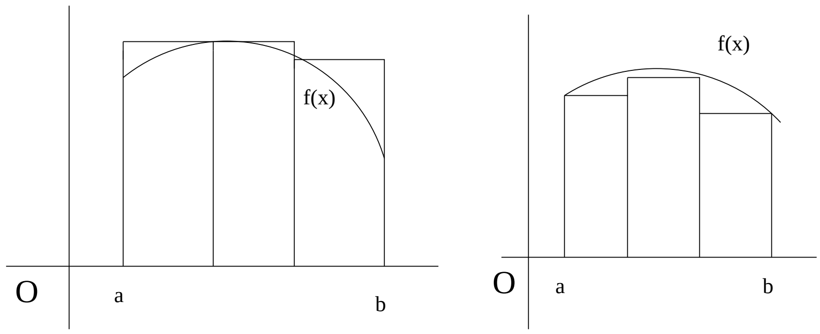

Figure 2

Since A is between the total areas of the inscribed and superscribed triangles, we have

$$nr^2 sin(\pi/n)cos(\pi/n) \leq A \leq nr^2 tan(\pi/n).$$

By sandwich theorem, $A = \pi r^2$. **We will use this idea to define and evaluate the area of the region under a graph of a function.**

Suppose f is a non-negative function defined on the interval $[a, b]$. We first subdivide the interval into a finite number of subintervals. Then we squeeze the area of the region under the graph of f between the areas of the inscribed and superscribed rectangles constructed over the subintervals as shown in Figure 2. **If** the total areas of the inscribed and superscribed rectangles converge to the same limit as we make the partition of $[a, b]$ finer and finer then the area of the region under the graph of f can be defined as this limit and f is said to be integrable.

Let us define whatever has been explained above formally.

The Riemann Integral

Let $[a, b]$ be a given interval. A *partition* P of $[a, b]$ is a finite set of points $x_0, x_1, x_2, \ldots, x_n$ such that $a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b$. We write $P = \{x_0, x_1, x_2, \ldots, x_n\}$.

If P is a partition of $[a, b]$ we write $\Delta x_i = x_i - x_{i-1}$ for $1 \leq i \leq n$. Let f be a bounded real valued function on $[a, b]$. For a partition P of $[a, b]$, we define

$$M_i = \sup\{f(x) : x_{i-1} \leq x \leq x_i\} \;\; and \;\; m_i = \inf\{f(x) : x_{i-1} \leq x \leq x_i\}.$$

$$U(P, f) = \sum_1^n M_i \Delta x_i \;\; and \;\; L(P, f) = \sum_1^n m_i \Delta x_i.$$

The numbers $U(P, f)$ and $L(P, f)$ are called *upper and lower Riemann sums* for the partition P (see Figure 2).

Since f is bounded, there exist real numbers m and M such that $m \leq f(x) \leq M$, for all $x \in [a, b]$. Thus for every partition P,

$$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a).$$

We define

$$\overline{\int_a^b} f dx = \inf U(P,f) \tag{1}$$

and

$$\underline{\int_a^b} f dx = \sup L(P,f) \tag{2}$$

(1) and (2) are called *upper and lower Riemann integrals* of f over $[a,b]$ respectively.

If the upper and lower integrals are equal, we say that f *is Riemann integrable* and we write $f \in \mathcal{R}$. In this case the common value of (1) and (2) is called the Riemann integral of f and is denoted by

$$\int_a^b f dx \quad \text{or} \quad \int_a^b f(x)dx.$$

Not every bounded function is Riemann integrable. For example the function

$$f(x) = 1 \;\; \textit{if } x \textit{ is rational and } 0 \textit{ otherwise}$$

is not Riemann integrable over any interval $[a,b]$ (Check this).

We now investigate the integrability of f. Throughout we will assume that f is a bounded real function on $[a,b]$.

Definition: A partition P_2 of $[a,b]$ is said to be finer than a partition P_1 if $P_2 \supset P_1$. In this case we say that P_2 is a *refinement* of P_1. Given two partition P_1 and P_2, the partition $P_1 \cup P_2 = P$ is called their common refinement.

The following theorem illustrates that refining partition increases lower terms and decreases upper terms.

Theorem 1 * : *Let P_2 be a refinement of P_1 then*

$$L(P_1,f) \le L(P_2,f) \tag{3}$$

and

$$U(P_2,f) \le U(P_1,f). \tag{4}$$

Proof : First we assume that P_2 contains just one more point than P_1. Let this extra point be $x^\star$. Suppose $x_{i-1} < x^\star < x_i$, where x_{i-1} and x_i are consecutive points of P_1. Let

$$\begin{aligned} w_1 &= \inf\{f(x) : \; x_{i-1} \le x \le x^\star\} \\ w_2 &= \inf\{f(x) : \; x^\star \le x \le x_i\} \end{aligned}$$

Then $w_1 \ge m_i$ and $w_2 \ge m_i$ where

$$m_i = \inf\{f(x) : \; x_{i-1} \le x \le x_i\}.$$

Then

$$L(P_2,f) - L(P_1,f) = w_1(x^\star - x_{i-1}) + w_2(x_i - x^\star) - m_i(x_i - x_{i-1}) \ge 0.$$

If P_2 contains k more points then we repeat this process $k-$times. (4) is analogously proved. (Prove it). $\square$

The geometric interpretation suggests that the lower integral is less than or equal to the upper integral. So the next result is also anticipated.

Corollary 2 $*$:

$$\overline{\int_a^b} f dx \geq \underline{\int_a^b} f dx$$

Proof: Let P_1, P_2 be two partitions and let P be their refinement. Then

$$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f).$$

Thus for any two partitions P_1 and P_2, we have $L(P_1, f) \leq U(P_2, f)$.
Fix P_2 and take sup over all P_1. Then

$$\underline{\int_a^b} f dx \leq U(P_2, f)$$

Now take inf over all P_2 to get the desired result. $\square$

We now present a necessary and sufficient condition for the existence of the integral of a bounded function. This criterion is analogous to the Cauchy criterion for the convergence of a sequence.

Theorem 3 * : (Riemann's criterion for integrability): *$f \in \mathcal{R}$ on $[a, b] \Leftrightarrow$ for every $\epsilon > 0$ there exists a partition P such that*

$$U(P, f) - L(P, f) < \epsilon \tag{5}$$

Proof: For any P, we have

$$L(P, f) \leq \underline{\int_a^b} f dx \leq \overline{\int_a^b} f dx \leq U(P, f).$$

Therefore (5) implies that

$$\overline{\int_a^b} f dx - \underline{\int_a^b} f dx < \epsilon, \quad \forall\ \epsilon\ >\ 0$$

Hence

$$\underline{\int_a^b} f dx = \overline{\int_a^b} f dx$$

i.e. $f \in \mathcal{R}$. Conversely, suppose $f \in \mathcal{R}$ and $\epsilon > 0$. Then there exist partitions P_1 and P_2 such that

$$U(P_2, f) - \int_a^b f dx < \epsilon/2 \text{ and } \int_a^b f dx - L(P_1, f) < \epsilon/2$$

Let P be the common refinement of P_1 and P_2. Then

$$U(P, f) - L(P, f) < \epsilon.$$

$\square$

We will apply the Riemann criterion for intergrabilty to prove the following two existence theorems.

Theorem 4: *If f is continuous on $[a, b]$ then $f \in \mathcal{R}$.*

Proof : Let $\epsilon > 0$. Since f is uniformly continuous, choose $\delta > 0$ such that $|s - t| < \delta \Rightarrow |f(s) - f(t)| < \epsilon$ for $s, t \in [a, b]$.

Let P be a partition of $[a, b]$ such that $\Delta x_i < \delta \ \forall \, i = 1, 2, \ldots, n$. Then

$$M_i - m_i \leq \epsilon \ \forall \, i = 1, 2, \ldots, n.$$

Hence

$$U(P, f) - L(P, f) = \sum_1^n (M_i - m_i)\Delta x_i \leq \epsilon(b - a).$$

This implies that $f \in \mathcal{R}$. $\square$

Theorem 5: *If f is a monotone function then $f \in \mathcal{R}$.*

Proof : Suppose f is monotonically increasing (the proof is similar in the other case.) Choose a partition P such that $\Delta x_i = \frac{b-a}{n}$. Then $M_i = f(x_i)$ and $m_i = f(x_{i-1})$. Therefore

$$\begin{aligned} U(P, f) - L(P, f) &= \frac{b-a}{n} \sum_1^n [f(x_i) - f(x_{i-1})] \\ &= \frac{b-a}{n} [f(b) - f(a)] \\ &< \epsilon \qquad \text{for large } n. \end{aligned}$$

Hence $f \in \mathcal{R}$. $\square$

We now see that if f is integrable, then the integral of f is the limit of the Riemann sums.

Definition: Let $f : [a, b] \to \mathbb{R}$ and let $P = \{x_0, x_1, ...x_n\}$ be a partition of $[a, b]$. Let $c_k \in [x_{k-1}, x_k], k = 1, 2, ...n$. Then a *Riemann sum* for f (corresponding to the partition P and the intermediate points c_k) is

$$S(P, f) = \sum_{k=1}^n f(c_k)\Delta x_k.$$

The norm of P is defined by

$$\gamma(P) = max_{1 \leq i \leq n} \Delta x_i.$$

Theorem 4: *Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable. Then*

$$\lim_{\gamma(P) \to 0} S(P, f) = \int_a^b f(x)dx$$