Handout 2, Properties of Continuous Functions on a Closed Interval.

For $a, b \in \mathbb{R}$, let
$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$
We call $[a, b]$ a closed interval of $\mathbb{R}$.

We will first see some properties of real number system.

The following theorem, called *nested interval theorem,* is a famous result about the set of real numbers.

Theorem 2.1 * : For each n, let $I_n = [a_n, b_n]$ be a (nonempty) bounded interval of real numbers such that
$$I_1 \supset I_2 \supset \cdots \supset I_n \supset I_{n+1} \supset \cdots$$
and $\lim\limits_{n\to\infty}(b_n - a_n) = 0$. Then $\bigcap\limits_{n=1}^{\infty} I_n$ contains only one point.

Proof: Note that the sequences (a_n) and (b_n) are respectively increasing and decreasing sequences; moreover both are bounded. Hence both converge, say $a_n \to a$ and $b_n \to b$. Then $a_n \leq a$ and $b \leq b_n$ $\forall$ $n \in N$. Since $b - a = \lim(b_n - a_n) = 0$. Thus $a = b$. Since $a_n \leq b_n$ $\forall$ n; this implies that $a \in \bigcap\limits_{n=1}^{\infty} I_n$. Clearly if $x \neq a$ then x does not belong to $\bigcap\limits_{n=1}^{\infty} I_n$.

The following two theorems are consequences of nested interval theorem.

Theorem 2.2 * : Every Cauchy sequence is convergent.

Proof: Since every Cauchy sequence is bounded, for an arbitrary n, we define
$$a_n = \inf_{k>n} x_k \quad \text{and} \quad b_n = \sup_{k>n} x_k.$$
Then the intervals $[a_n, b_n]$ form a nested family:
$$[a_{n+1}, b_{n+1}] \subset [a_n, b_n].$$
Since $\{x_n\}$ is a Cauchy sequence, $b_n - a_n \to 0$ as $n \to \infty$. Hence by the nested interval theorem, there exists x such that $x \in \bigcap\limits_{n=1}^{\infty} [a_n, b_n]$. This x is the limit of $\{x_n\}$ (verify !).

Theorem 2.3 * **(Bolzano-Weierstrass theorem) :** Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof: (Sketch). Let $\{x_n\}$ be a bounded sequence such that the set $\{x_1, x_2, \cdots\} \subset [a, b]$. Divide this interval into two equal parts. Let I_1 be that interval which contains an infinite number of elements of $\{x_n\}$. Let x_{n_1} be one of the elements belonging to the interval I_1. Divide I_1 into two equal parts and let I_2 be that interval which contains an infinite number of elements. Choose a point x_{n_2} in I_2 such that $n_2 > n_1$. Keep dividing the intervals I_k, to generate I_k's and x_{n_k}'s. By nested interval theorem $\bigcap\limits_{k=1}^{\infty} I_k = \{x\}$, for some $x \in [a, b]$. Then the subsequence $\{x_{n_k}\}$ converges to x. $\square$

We will now see some properties of continuous functions on a closed interval.

Theorem 2.4 : If a function f is continuous on a closed interval $[a, b]$ then it is bounded on $[a, b]$.

Proof: Suppose that f is not bounded on $[a, b]$. Then for each natural number n there is a point $x_n \in [a, b]$ such that $|f(x_n)| > n$, for all $n = 1, 2, \ldots$. Since (x_n) is a bounded sequence, by Bolzano-Weierstrass theorem it has a convergent subsequence say $x_{n_k} \to x_0 \in [a, b]$. By the continuity of f, we have $f(x_{n_k}) \to f(x_0)$. This contradicts the assumption that $|f(x_n)| > n \; \forall \; n$. Hence$f$ is bounded on $[a, b]$. $\Box$

We remark that if a function is continuous on an open interval (a, b) or on a semi-open interval of the type $(a, b]$ or $[a, b)$, then it is not necessary that the function has to be bounded. For example, consider the continuous function $\frac{1}{x}$ on $(0, 1]$.

Let f be defined on a subset S of $\mathbb{R}$. An element $x_0 \in S$ is called a *maximum* for f on S if $f(x_0) \geq f(x) \; \forall \; x \in S$. Similarly, x_0 is called a *minimum* for f on S if $f(x_0) \leq f(x) \; \forall \; x \in S$.

The following theorem provides an existence criterion for the maximum and minimum.

Theorem 2.5 : Let f be continuous on $[a, b]$. Then there exist $x_0, y_0 \in [a, b]$ such that x_0 is a maximum for f on $[a, b]$ and y_0 is a minimum for f on $[a, b]$.

Proof: By the previous theorem f is bounded on $[a, b]$. Let M be the least upper bound of $f([a, b])$ $(:= \{f(x) : x \in [a, b]\})$. Then there exists a sequence $\{f(x_n)\}$ in $f([a, b])$ such that $f(x_n) \to M$. Since $\{x_n\}$ is a bounded sequence in $[a, b]$, it has a convergent subsequence, say $x_{n_k} \to x_0 \in [a, b]$. By the continuity of f we have $f(x_0) = M$. Hence x_0 is a maximum for f on $[a, b]$.

The proof for the minimum is similar. $\Box$

Consider a function $f : [-1, 1] \to \mathbb{R}$ such that f is continuous and satisfies $f(-1) < 0$ and $f(1) > 0$. Intuitively, we feel that the graph of f should cross the x-axis between -1 and 1; i.e., there is some $x_0 \in [-1, 1]$ such that $f(x_0) = 0$. This motivates us to state the following theorem.

Theorem 2.6 * **(Intermediate Value Property) :** Let f be continuous on $[a, b]$, and let $f(a) < s < f(b)$. Then there exists x such that $a < x < b$ and $f(x) = s$.

Proof: Let $S = \{x \in [a, b] : f(x) \leq s\}$. Since $a \in S$, we have $S \neq \emptyset$. Let c be the least upper bound of S. We claim that $f(c) = s$. Since c is the least upper bound of S, there exists a sequence $\{x_n\}$ from S such that $x_n \to c$. By continuity of f, $f(x_n) \to f(c)$. Since $f(x_n) \leq s \; \forall \; n$, we have $f(c) \leq s$. Note that $b > c$. Consider the sequence $y_n = c + (b - c)/n$. As $y_n \to c$, we get $f(y_n) \to f(c)$. Since $f(y_n) > s \; \forall \; n$, we have $f(c) \geq s$. It follows that $f(c) = s$.

Theorem 2.7 * : Let f be one-one, on to and continuous on $[a, b]$. Then f^{-1} is also continuous.

Proof:(Sketch) First note that $f([a, b])$ is an interval. Let $\{y_n\}$ be a sequence in $f([a, b])$ such that $y_n \to y_0$. Choose x_n, x_0 in $[a, b]$ such that $f(x_n) = y_n$ and $f(x_0) = y_0$. Using the Bolzano-Weierstrass theorem and the continuity of f one can show that $x_n \to x_0$. This proves the continuity of f^{-1}.