Assignment- 7 : Integration

1. If f is a bounded function such that $f(x) = 0$ except at a point $c \in [a, b]$, then show that f is integrable on $[a, b]$ and that $\int_a^b f = 0$.

2. Define $f : [0, 2] \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1, & -1 \le x \le 0 \\ 1, & 0 < x \le 1 \end{cases}$$

   Is the function continuous on $[-1, 1]$? Is the function Riemann integrable?

3. Suppose that f is a continuous function on $[a, b]$ such that $f(x) \ge 0, \forall x \in [a, b]$. Show that $\int_a^b f = 0$ if and only if $f(x) = 0\ \forall x \in [a, b]$.

   Compare this with the function f in Question 2. What is $\int_{-1}^1 f(x)dx$? Does this contradict the above statement? Explain.

4. Let $f : [0,\ 1] \longrightarrow \mathbb{R}$ such that

   $f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$.

   Show that f is integrable on $[0,\ 1]$ and $\int\limits_0^1 f(x)dx = 0$.

5. If f and g are continuous functions on $[a,\ b]$ and if $g(x) \ge 0$ for $a \le x \le b$, then show that there exists $c \in [a,\ b]$ such that $\int\limits_a^b f(x)g(x)dx = f(c)\int\limits_a^b g(x)dx$.

   (This result is sometimes called the second mean value theorem for integrals. The special case $g = 1$ yields the first mean value theorem for integrals.)

6. Let $g_n(y) = \begin{cases} \frac{ny^{n-1}}{1+y} & \text{if } 0 \le y < 1 \\ 0 & \text{if } y = 1 \end{cases}$. Then prove that

   $\lim\limits_{n\longrightarrow\infty} \int\limits_0^1 g_n(y)dy = \frac{1}{2}$ whereas $\int\limits_0^1 \lim\limits_{n\longrightarrow\infty} g_n(y)dy = 0.$

7. Test the convergence/divergence of the following improper integrals:

   (a) $\int\limits_0^1 \frac{dx}{\log(1+\sqrt{x})}$ (b) $\int\limits_0^1 \frac{dx}{x-\log(1+x)}$ (c) $\int\limits_0^1 \frac{\log x}{\sqrt{x}}$ (d) $\int\limits_0^1 \sin(1/x)dx.$

   (e) $\int\limits_1^\infty \frac{\sin(1/x)}{x}dx$ (f) $\int\limits_0^\infty e^{-x^2}dx.$

8. In each case, determine the values of p for which the following improper integrals absolutely converge

(a) $\int\limits_0^\infty \frac{1-e^{-x}}{x^p}dx$ (b) $\int_0^\infty \frac{t^{p-1}}{1+t}dt$ (c) $\int\limits_1^\infty \frac{\sin x}{x^p}dx.$

9. Show that the integrals $\int\limits_0^\infty \frac{\sin^2 x}{x^2}dx$ and $\int\limits_0^\infty \frac{\sin x}{x}dx$ converge. Further, prove that $\int\limits_0^\infty \frac{\sin^2 x}{x^2}dx = \int\limits_0^\infty \frac{\sin x}{x}dx.$

10. Show that $\int\limits_0^\infty \frac{x \log x}{(1+x^2)^2}dx = 0.$

11. Prove that improper integral $\int_1^\infty \frac{\sin x}{x^p}dx$ converges conditionally for $0 < p \leq 1$ and absolutely for $p > 1$.

12. Show that $\int\limits_0^s \frac{1+x}{1+x^2}dx$ and $\int\limits_{-s}^0 \frac{1+x}{1+x^2}dx$ do not approach a limit as $s \longrightarrow \infty$. However $\lim\limits_{s \longrightarrow \infty} \int\limits_{-s}^s \frac{1+x}{1+x^2}dx$ exists.