Assignment - 6 : Series

1. Let $a_n \geq 0$. Then both the series $\sum\limits_{n\geq 1} a_n$ and $\sum\limits_{n\geq 1} \frac{a_n}{a_n+1}$ converge or diverge together.

2. Prove that $\sum (a_n - a_{n+1})$ converges if and only if the sequence a_n converges. Use this to decide the convergence/divergence of the following series:

   (1) $\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)}$ (2) $\sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2}$

3. **Limit comparison test:** Suppose that $a_n, b_n > 0$ for all $n \in \mathbb{N}$.

   (a) If $\lim\limits_{n\longrightarrow\infty} \dfrac{a_n}{b_n} = c > 0$ then $\sum\limits_{n\geq 1} a_n$ and $\sum\limits_{n\geq 1} b_n$ both converge or both diverge.

   (b) If $\lim\limits_{n\longrightarrow\infty} \frac{a_n}{b_n} = 0$ and $\sum\limits_{n\geq 1} b_n$ converges then $\sum\limits_{n\geq 1} a_n$ converges.

   (c) If $\lim\limits_{n\longrightarrow\infty} \frac{a_n}{b_n} = \infty$ and $\sum\limits_{n\geq 1} b_n$ diverges then $\sum\limits_{n\geq 1} a_n$ diverges.

4. In each of the following cases, discuss the convergence/divergence of the series $\lim\limits_{n\geq 1} a_n$ where a_n equals:

   (a) $1 - n\sin\frac{1}{n}$ (b) $\frac{1}{n}\log(1+\frac{1}{n})$ (c) $1-\cos\frac{1}{n}$

   (d) $2^{-n-(-1)^n}$ (e) $\left(1+\frac{1}{n}\right)^{n(n+1)}$ (f) $\frac{n\ln n}{2^n}$

5. Let $\sum\limits_{n\geq 1} a_n$ and $\sum\limits_{n\geq 1} b_n$ be series of positive terms satisfying $\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n}$ for all $n \geq N$. Show that if $\sum\limits_{n\geq 1} b_n$ converges then $\sum\limits_{n\geq 1} a_n$ also converges.

6. Test the series $\sum\limits_{n\geq 1} \frac{n^{n-2}}{e^n n!}$ for convergence.

7. Let $\{a_n\}$ be a decreasing sequence, $a_n \geq 0$ and $\lim\limits_{n\longrightarrow\infty} a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Show that $\sum\limits_{n\geq 1}(-1)^n b_n$ converges.

8. Determine the values of x for which the following series converges:

   (a) $\sum\limits_{n\geq 1} \frac{x^{2n}}{n^2 3^n}$ (b) $\sum\limits_{n\geq 1} \frac{n^3}{3^n} x^n$ (c) $\sum\limits_{n\geq 1} \frac{(2n)!}{(2^n n!)^2} \frac{x^{2n+1}}{2n+1}$

9. Find the Taylor series at 0 for each of the following functions and also the values of x for which the corresponding series converges:

   (a) $f(x) = \frac{1}{x-a}$ $a \neq 0$ (b) $f(x) = \frac{1}{\sqrt{1-x}}$