Assignment - 5 : Mean Value Theorem, Taylor's Theorem

1. Prove that $\log(1+x) > \frac{x}{1+x}$ for all $x > 0$.
2. Prove that
   (a) $1 - \frac{x^2}{2!} < \cos x < 1 - \frac{x^2}{2} + \frac{x^4}{4!}$ for $x \neq 0$.
   (b) $x - \frac{x^3}{3!} < \sin x$ for $x > 0$.
   (c) $\cos x < 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$ for $x \neq 0$.
   (d) $\sin x < x - \frac{x^3}{3!} + \frac{x^5}{5!}$ for $x > 0$.
3. Let g be a bounded and differential function. Show that if $\lim_{t\to\infty} g'(t)$ exists then the limit has to be zero.
4. For $x > -1, x \neq 0$ prove that
   (a) $(1+x)^\alpha > 1 + \alpha x$, whenever $\alpha < 0$, or $\alpha > 1$
   (b) $(1+x)^\alpha < 1 + \alpha x$, whenever $0 < \alpha < 1$.
5. Let $f : [a,b] \longrightarrow \mathbb{R}$ be continuously differentiable on $[a,b]$ and twice differentiable on (a,b), and suppose that $f(a) = f'(a) = f(b) = 0$. Prove that there exists $x_1 \in (a,b)$ such that $f''(x_1) = 0$.
6. Let f be continuous on $[a,b]$, $a > 0$ and differentiable on (a,b). Prove that there exists $c \in (a,b)$ such that
$$\frac{bf(a) - af(b)}{b-a} = f(c) - cf'(c).$$
7. Suppose f is a three times differentiable function on $[-1,1]$ such that $f(-1) = 0$, $f(1) = 1$ and $f'(0) = 0$. Using Taylor's theorem prove that $f'''(c) \geq 3$ for some $c \in (-1,1)$.
8. Using Taylor's theorem, for any $k \in \mathbb{N}$ and for all $x > 0$, show that
   $x - \frac{1}{2}x^2 + \ldots \frac{1}{2k}x^{2k} < \log(1+x) < x - \frac{1}{2}x^2 + \ldots \frac{1}{2k+1}x^{2k+1}$.
9. Find $\lim_{x\longrightarrow 5}(6-x)^{\frac{1}{x-5}}$ and $\lim_{x\longrightarrow\infty} \frac{x - \sin x}{2x + \sin x}$.
10. Sketch the graphs of $f(x) = x^3 - 6x^2 + 9x + 1$ and $f(x) = \frac{x^2}{x^2-1}$.