Assignment - 4 : Derivatives, Maxima, Minima, Rolle's Theorem

1. Examine the function $f(x) = \begin{cases} x \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ for differentiability.

2. Show that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not continuous at $x = 0$. Thus, a function that is differentiable at every point of $\mathbb{R}$ need not have a continuous derivative $f'(x)$.

3. Show that the function $f(x) = \begin{cases} x^2 \sin \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at all $x \in \mathbb{R}$. Also show that the function $f'(x)$ is not bounded on the interval $[-1, 1]$.

4. Let $f(0) = 0$ and $f'(0) = 1$. For a positive integer k show that
   $\lim\limits_{x \to \infty} \frac{1}{x}(f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + \cdots + f(\frac{x}{k}) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

5. Prove that if $f : \mathbb{R} \longrightarrow \mathbb{R}$ is an *even function* ($f(-x) = f(x)$ for all $x \in \mathbb{R}$) and has a derivative at every point, then the derivative f' is an *odd function* ($f(-x) = -f(x)$ for all $x \in \mathbb{R}$).

6. Show that there is exactly one real value of $x \in (0, 1)$ that satisfies $(1-x)\cos x = \sin x$.

7. Let f and g be functions, continuous on $[a,\ b]$, differentiable on $(a,\ b)$, and let $f(a) = f(b) = 0$. Prove that there is a point $c \in (a,\ b)$ such that $g'(c)f(c) + f'(c) = 0$.

8. Suppose f is continuous on $[a,\ b]$, differentiable on $(a,\ b)$, and satisfies $f^2(a) - f^2(b) = a^2 - b^2$. The show that the equation $f'(x)f(x) = x$ has at least one root in $(a,\ b)$.

9. Prove that the equation $x^{13} + 7x^3 - 5 = 0$ has exactly one real root.