Assignment - 3 : Continuous Functions, Intermediate Value Property

1. Determine the points of continuity for the functions $f : [0,1] \longrightarrow [0,1]$ defined by

   $a)\;\; f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ $\qquad b)\;\; f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ x & \text{if } x \text{ is irrational} \end{cases}$

   $$c)\;\; f(x) = \begin{cases} 4(x-\frac{1}{2})^2 & \text{if } x \text{ is rational} \\ 1-4(x-\frac{1}{2})^2 & \text{if } x \text{ is irrational} \end{cases}.$$

2. Let
   $$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
   Prove that f is continuous at every point except for the point $x = 0$.

3. Let $f : [a,b] \longrightarrow \mathbb{R}$ be any function satisfying

   (a) $f(x)$ is rational for every x that is irrational, and
   (b) $f(x)$ is irrational for every x that is rational.

   Prove that f cannot be a continuous function.

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function and let $c \in \mathbb{R}$. Show that if $x_0 \in \mathbb{R}$ is such that $f(x_0) > c$, then there exists a $\delta > 0$ such that $f(x) > c$ for all $x \in (x_0 - \delta, x_0 + \delta)$.

5. Prove that if a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is also one-one then either f is a strictly increasing function or f is a strictly decreasing function.

6. Let $f : [1,3] \longrightarrow \mathbb{R}$ be a continuous function. Prove that there exists real numbers $x_1, x_2 \in [1,3]$ such that
   $$x_2 - x_1 = 1 \quad \text{and} \quad f(x_2) - f(x_1) = \frac{1}{2}\big(f(3) - f(1)\big).$$

7. Prove that the equation $(1-x)\cos x = \sin x$ has at least one solution in $(0,1)$.

8. Let $p(x)$ be the polynomial
   $$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad a_n \neq 0.$$
   Prove that

   (a) if n is odd, then $p(x)$ has at least one real root.
   (b) if a_n and a_0 have opposite signs, then $p(x)$ has at least one positive root, and in addition, if n is even, $n \neq 0$, then $p(x)$ has a negative root as well.