Assignment - 2 : Sequences

1. Prove if $x_n \to 0$ and $y_n \to 0$, then $x_n y_n \to 0$.

2. Show that a sequence $\{x_n\}$ does not converge to zero if and only if there exists $\epsilon > 0$ such that $|x_n| \geq \epsilon$ for infinitely many n's.

3. True or false, explain:
   If $x_n y_n \to 0$, then either $x_n \to 0$ or $y_n \to 0$. Compare this question with Q.1.

4. Let x be any real number. Then prove that there exist sequences $\{x_n\} \longrightarrow x$ such that

   (a) all the x_n's are rational numbers,

   (b) all the x_n's are irrational numbers,

   (c) the x_n's are a mixture of rational or irrational depending on our choice.

5. Let $\{x_n\}$ be a sequence of strictly positive real numbers such that $\lim\limits_{n \longrightarrow \infty} \frac{x_{n+1}}{x_n} = \ell$. Then prove the following:

   (a) if $\ell < 1$ then $\lim\limits_{n \longrightarrow \infty} x_n = 0$,

   (b) if $\ell > 1$ then $\lim\limits_{n \longrightarrow \infty} x_n = \infty$, and

   (c) if $\ell = 1$ then give example of sequences to show that both conclusions can hold.

6. Investigate the convergence of the following sequences:

   (a) $x_n = \frac{1}{1^2+1} + \frac{1}{2^2+2} + \cdots + \frac{1}{n^2+n}$,

   (b) $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}$,

   (c) $x_n = (n+1)^\alpha - n^\alpha$ for some $\alpha \in (0,1)$,

   (d) $x_n = n y^{n-1}$ for some $y \in (0,1)$,

   (e) $x_n = \frac{n^s}{(1+p)^n}$ for some $s > 0$ and $p > 0$,

   (f) $x_n = \frac{1-2+3-4+\cdots+(-1)^{n-1}n}{n}$,

   (g) $x_n = \frac{2^n}{n!}$, $x_n = \frac{n^2}{n!}$.

   (h) $x_1 = \sqrt{3}$ and $x_n = \sqrt{3 + x_{n-1}}$ for $n > 1$,

   (i) $x_1 = 1$, $x_2 = 2$ and $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ for $n > 2$.

7. Let $a, x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Prove that the sequence $\{x_n\}$ converges to $\sqrt{a}$. *These sequences are used in the numerical calculation of* $\sqrt{a}$.

8. Prove that the sequence $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots$ converges. Find the limit.

9. Suppose that $0 < \alpha < 1$ and that $\{x_n\}$ is a sequence which satisfies one of the following conditions

   (a) $|x_{n+1} - x_n| \leq \alpha^n, \qquad n = 1, 2, 3, \ldots$

   (b) $|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|, \qquad n = 1, 2, 3, \ldots.$

   Then prove that $\{x_n\}$ satisfies the Cauchy criterion.

   Whenever you use this result, you have to show that the number α that you get, satisfies $0 < \alpha < 1$ and that α does not depend on n. The condition $|x_{n+2} - x_{n+1}| \leq |x_{n+1} - x_n|$ does not guarantee the convergence of $\{x_n\}$. Give examples.

10. Let $\{x_n\}$ be defined by $x_1 = 1, x_2 = 2$ and $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})\ \forall n \geq 3$.

    (a) Show that $1 \leq x_n \leq 2\ \forall n \in \mathbb{N}$. Draw a picture to see what is going on; Is the sequence monotone ? (Think of mid points).

    (b) Show that $\{x_n\}$ is Cauchy. (Hint: Use Q9).

    (c) Find the limit.