Assignment- 13 : Triple Integrals, Surface Integrals

1. Suppose that the integral $\int_0^1 \int_0^{1-x} \exp^{y/(x+y)}\, dydx$ exists. Evaluate it.

2. Integrate $z \exp^{x^2+y^2}\, dxdydz$ over the cylinder $x^2 + y^2 \leq 4,\ 2 \leq z \leq 3$.

3. Evaluate the integral $\iiint_W \frac{dzdydx}{\sqrt{1+x^2+y^2+z^2}}$; where W is the ball $x^2 + y^2 + z^2 \leq 1$.

4. Find the line integral of the vector field $F(x, y, z) = y\vec{i} - x\vec{j} + \vec{k}$ along each of the following paths joining $(1, 0, 0)$ to $(1, 0, 1)$.

   (a) $\mathbf{c}_1(t) = (\cos t, \sin t, \frac{t}{2\pi}),\ \ 0 \leq t \leq 2\pi$

   (b) $\mathbf{c}_2(t) = (\cos t^3, \sin t^3, \frac{t^3}{2\pi}),\ \ 0 \leq t \leq \sqrt[3]{2\pi}$

   (c) $\mathbf{c}_3(t) = (\cos t, -\sin t, \frac{t}{2\pi}),\ \ 0 \leq t \leq 2\pi$.

5. Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 1$ lying above the elliptic region $x^2 + \frac{y^2}{a^2} \leq 1$; for $0 < a \leq 1$.

6. What is the integral of the function $x^2 z$ taken over the entire surface of a right circular cylinder of height h which stands on the circle $x^2 + y^2 = a^2$. What is the integral of the given function taken throughout the volume of the cylinder.

7. Compute $\iint_S xy\ dS$, where S is the surface of the cone $x = r\cos t,\ y = r\sin t, ; z = r$ for $0 \leq r \leq 1$ and $0 \leq t \leq 2\pi$.

8. Find the area of the surface defined by $x + y + z = 1,\ x^2 + 2y^2 \leq 1$.

9. Let $\mathbf{r}(x, y, z) = (x, y, z)$ and $r = \|\mathbf{r}\| = \sqrt{x^2 + y^2 + z^2}$. If $r \neq 0$, prove that

$$\nabla\left(\frac{1}{r}\right) = \frac{-\mathbf{r}}{r^3} \quad \text{and} \quad \nabla^2\left(\frac{1}{r}\right) = \nabla \cdot \left(\nabla\left(\frac{1}{r}\right)\right) = 0.$$

10. Let $F = xy^2\vec{i} + (y + x^2)\vec{j}$. Integrate $(\nabla \times F) \cdot \vec{k}$ over the region in the first quadrant bounded by the curves $y = x^2$ and $y = x$.