Assignment - 10 : Functions of several variables

1. Examine the following functions for continuity at the point $(0,0)$, where $f(0,0) = 0$ and $f(x,y)$ for $(x,y) \neq (0,0)$ is given by

   $i)\ \frac{xy}{\sqrt{x^2+y^2}} \qquad ii)\ \frac{xy}{x^2+y^2} \qquad iii)\ \frac{x^4-y^2}{x^4+y^2} \qquad iv)\ \frac{x^2y}{x^4+y^2}.$

2. Consider the functions $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

   $$(i)\ f(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ or if } y = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (ii)\ f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if} (x,y) = (0,0) \end{cases}$$

   Show that the functions satisfy the following:

   (a) The iterated limits $\lim_{x \longrightarrow 0} \left[\lim_{y \longrightarrow 0} f(x,y) \right]$ and $\lim_{y \longrightarrow 0} \left[\lim_{x \longrightarrow 0} f(x,y) \right]$ exist and equals 0;

   (b) $\lim_{(x,y) \longrightarrow (0,0)} f(x,y)$ does not exist;

   (c) $f(x,y)$ is not continuous at $(0,0)$; but

   (d) the partial derivatives exist at $(0,0)$.

3. Let $f(x,y) = xy\frac{x^2-y^2}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and 0, otherwise. Prove that

   (a) $f_x(0,y) = -y$ and $f_y(x,0) = x$ for all x and y;

   (b) $f_{xy}(0,0) = -1$ and $f_{yx}(0,0) = 1$ and (c) $f(x,y)$ is differentiable at $(0,0)$.

4. Let $f(x,y) = (x^2+y^2)\sin\frac{1}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and 0, otherwise. Show that f is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives are not continuous at $(0,0)$.

5. Let $f(x,y)$ be defined in $S = \{(x,y) \in \mathbb{R}^2 : a < x < b,\ c < y < d\}$. Suppose that the partial derivatives of f exist and are bounded in S. Then show that f is continuous in S.

6. Consider the functions defined by

   $$(i)\ f(x,y) = \begin{cases} \frac{y^3}{x^2+y^2} & \text{if } (x,y) \neq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (ii)\ \begin{cases} \frac{y}{|y|}\sqrt{x^2+y^2} & \text{if } y \neq 0 \\ 0 & \text{otherwise} \end{cases}.$$

   Prove that in each case the functions f are continuous at $(0,0)$, they have all directional derivatives at $(0,0)$ but they are not differentiable at $(0,0)$.

7. Suppose f is a function with $f_x(x,y) = f_y(x,y) = 0$ for all (x,y). Then show that $f(x,y) = c$, a constant.

8. Let $f(x,y) = \frac{y}{|y|}\sqrt{x^2+y^2}$ if $y \neq 0$ and $f(x,y) = 0$ if $y = 0$. Show that f is continuous at $(0,0)$, it has all directional derivatives at $(0,0)$ but it is not differentiable at $(0,0)$.

9. Let $f(x,y) = \frac{1}{2}\Big(\big|\,|x| - |y|\,\big| - |x| - |y|\Big)$. Is f continuous at $(0,0)$? Which directional derivatives of f exist at $(0,0)$? Is f differentiable at $(0,0)$?

10. Find the equation of the surface generated by the normals to the surface $x + 2yz + xyz^2 = 0$ at all points on the z-axis.