Assignment - 1 : Real numbers

1. For any $a, b \in \mathbb{R}$, prove the following inequalities:
   (1) $|ab| = |a||b|$ (2) $|a+b| \leq |a| + |b|$ (Triangle Inequality)
   (3) $|a-b| \leq |a| + |b|$ (4) $|a| - |b| \leq |a-b|$
   (5) $\big||a| - |b|\big| \leq |a-b|$.

2. Prove that for every $x, y \in \mathbb{R}$,
$$\max(x,y) = \frac{x+y+|x-y|}{2} \quad \text{and} \quad \min(x,y) = \frac{x+y-|x-y|}{2}.$$

3. Prove the Schwartz inequality
   $x_1y_1 + x_2y_2 \leq \sqrt{{x_1}^2 + {x_2}^2}\sqrt{{y_1}^2 + {y_2}^2}$
   by first proving that $({x_1}^2 + {x_2}^2)({y_1}^2 + {y_2}^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2$.

4. Let x and n be positive integers. Show that if $\sqrt[n]{x}$ is rational, then it must be an integer.

5. Suppose $a \in \mathbb{R}$ satisfies $0 \leq a < \epsilon$ for all $\epsilon > 0$. Then prove that $a = 0$.

6. Let A be a nonempty subset of $\mathbb{R}$ and let α be a real number. Prove that $\alpha = \sup A$ if and only if
   (a) $\alpha \geq a$ for every $a \in A$, and
   (b) for any $\epsilon > 0$ there exists some $a_0 \in A$ such that $\alpha - \epsilon < a_0$.

7. Let $A = \{x \in \mathbb{R} : x^2 - 3x + 2 \leq 0\}$. Find $\sup A$ and $\inf A$.

8. Find the supremum and infimum of the following sets:
   a) $\left\{\frac{m}{n} : m, n \in \mathbb{N}, m < 2n\right\}$ *b)* $\left\{\frac{m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z}\right\}$.

9. Prove that between any two real numbers there is an irrational number in $\mathbb{R}$.