On the Uniqueness and Multiplicity of positive solutions to an elliptic spectral problem with concave and convex nonlinearity

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Abstract

Population models with a diffusive spread of a population following a nonlinear growth pattern have a long history. Selecting, for the growth, a mixture of a concave nonlinearity at low population densities and a convex one at high densities, in the late 1970s and early 1980s a number of researchers have developed powerful methods of partial differential equations and functional analysis to treat such problems assuming a *uniform, space-idependent* growth throughout the domain of the habitat. A typical result was the possibility of *two positive equilibria* of a lower and a higher population density, pointwise ordered throughout the habitat. In our presentation we will construct a similar pair of equilibria in a model with *space-dependent growth*: concave in one subdomain and convex in the other one, linear on the boundary (or region) between the two subdomains.

We will discuss the question of ***existence*** and ***multiplicity*** of *positive solutions* to the semilinear elliptic Dirichlet problem

$$-\Delta u = \lambda\, u(x)^{q(x)-1} + f(x,u(x)) \quad \text{for } x\in\Omega\,; \qquad u=0 \quad \text{on } \partial\Omega\,, \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with the boundary of class $C^{1,\alpha}$, $\lambda\in\mathbb{R}$ a spectral parameter, and $f(x,u) = |u|^{r-1}\,u$ is a **signed r-power** ($r>0$) of the unknown function of (a positive variable) $u\in(0,\infty)$ which depends on the point $x\in\Omega$; $r = q(x)-1$, for instance.

We will briefly present basic methods for treating the semilinear problem (0.1) with a ***convex*** and ***concave*** nonlinear reaction $f(x,\,\cdot\,) : s \longmapsto |s|^{q(x)-2}s : \mathbb{R}_+ \subset \mathbb{R} \to \mathbb{R}$ which (for $s\geq 0$) is ***convex*** in a nonempty open subset $\Omega_+ \stackrel{\text{def}}{=} \{x\in\Omega : q(x)>2\}$ and ***concave*** in another nonempty open subset $\Omega_- \stackrel{\text{def}}{=} \{x\in\Omega : q(x)<2\}$ of a bounded domain $\Omega\subset\mathbb{R}^N$. Here, $\lambda\in\mathbb{R}_+$ is a nonnegative spectral parameter which decides about the existence and multiplicity of positive weak solutions (at least two) to problem (0.1) in case we take $f\equiv 0$. Our main contribution is a method how to handle the interplay between *convex* and *concave* nonlinearities in two disjoint nonempty open subsets of a domain Ω (connected in $\mathbb{R}^N$), as opposed to the classical works assuming a nonlinearity $f(s)$ being *concave* for small values of $s\in\mathbb{R}_+$ and *convex* for large $s\in\mathbb{R}_+$, uniformly in Ω.

The Dirichlet Laplace operator being linear, we observe that the elliptic equation in problem (0.1) is ***convex*** (***concave***, respectively) at a given point $x\in\Omega$, provided $q(x)\geq 2$ ($1<q(x)\leq 2$). This is a typical problem with variable powers (exponents).

Finally, if time permits, we will discuss also the classical question of ***uniqueness*** for a related problem with the $p(x)$-Laplacian provided $q(x)\leq \text{const}_1 < \text{const}_2 \leq p(x)$ holds for all $x\in\Omega$, and the ***multiplicity*** of large solution branches bifurcating from infinity as $\lambda \searrow 0+$.

Running head: A semilinear elliptic problem with a convex/concave nonlinearity

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space-dependent exponent; convex / concave nonlinearity;
positivity and Hopf's boundary point lemma

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