Department of Mathematics & Statistics

Ph.D admission written test

Time: 90 Minutes **July 13, 2017**

Total Marks: 105

NAME:____

Instructions

1. Write your name in **CAPITAL** letters.
2. We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}[i]$ the set of natural numbers, integers, rational numbers, real numbers, complex numbers and Gaussian integers respectively.

   For $n \geq 1$, the set $\mathbb{Z}_n$ denotes the set $\mathbb{Z}/n\mathbb{Z}$ and S_n denotes the permutation group on n-symbols.

   We denote by $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, the unit disc in $\mathbb{C}$.
3. There is a provision for partial marking for questions in section 3.
4. There are three sections. The first section is True or false and the second section is fill in the blanks.
   - In the first section, every correct answer will be awarded 3 marks and a wrong answer will be awarded -3 marks.
   - In the second section every correct answer carries 3 marks.
5. The third section has one or more correct answers. In this section
   - each question has four choices.
   - if a wrong answer is selected in a question then that entire question will carry 0 marks.
   - the candidate gets full credit of 6 marks, only if he/she selects all the correct answers and no wrong answers. 3 marks will be awarded for an answer to a question if it is partially correct and a wrong answer is not selected.
6. These question-cum-answer-sheets must be returned to the invigilator before leaving the examination hall.

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1 True/False [24 marks]

1. There is a surjective group homomorphism from S_4 to S_3 but there is no surjective group homomorphism from S_5 to S_4.

2. Let $n \geq 2$ and $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation of rank 1. Then there exists a non-zero real number c such that $A^2 = cA$.

3. Let G be a finite group with a unique element x of order 2. Then the center $Z(G)$ is a group of even order.

4. Let $p : [a, b] \to \mathbb{R}$ be a continuous function such that $p(x) \leq C$ for all $x \in [a, b]$ and λ be an eigenvalue of the Sturm-Liouville equation

$$(x^2 u')' + p(x)u + \lambda u = 0 \text{ in } [a, b]$$
$$u(a) = u(b) = 0.$$

Then $\lambda \leq -C$.

5. Every solution to the equation $y'' + xy = 0$ has infinitely many zeros in $(0, \infty)$.

6. There exists a linear map $T : \mathbb{R}^8 \to \mathbb{R}^4$ such that $\text{Ker}(T) := \{(x_1, x_2, \ldots, x_8) \in \mathbb{R}^8 : x_1 + 2x_2 + \cdots + 8x_8 = 0\}$ and $\text{Im}(T) := \{(y_1, y_2, y_3, y_4) \in \mathbb{R}^4 : y_1 + y_2 + y_3 + y_4 = 0\}$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^3(1 - x^2)(1 + x)$ for $x \in \mathbb{R}$ and $\text{Graph}(f) := \{(x, f(x)) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$,the graph of f. Then $\text{Graph}(f)$ is homeomorphic to the interval $(-1, 1)$.

8. Consider the unit circle $\mathbb{S}^1 := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ in Euclidean plane $\mathbb{R}^2$. Let $f : \mathbb{S}^1 \to \mathbb{R}$ be a real valued continuous function. Then there exists $x \in \mathbb{S}^1$ such that $f(x) = f(-x)$.

2 Fill in the blanks [15 marks]

1. Let U and V are subspaces of $\mathbb{R}^3$ such that $U = \text{span}\,\{(1, 1, -1), (2, 3, -1), (3, 1, -5)\}$ and $V = \text{span}\{(1, 1, -3), (3, -2, -8), (2, 1, -3)\}$. Then $\dim(U \cap V)$ is __________.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $|f(z)| \to \infty$ as $|z| \to \infty$. Then f is a __________.

3. For an infinitely differentiable function f, $\alpha, \beta \in \mathbb{R}$ and $h > 0$, if the approximate second derivative

$$D_h f(x) = \frac{\alpha f(x) - 5f(x + h) + 4f(x + 2h) + \beta f(x + 3h)}{h^2}$$

yields error $f''(x) - D_h f(x) = Ch^k$ with a constant C independent of h, then k is ________ .

4. The number of solution(s) to the equation $u_x + u_y = 1$ such that $u(x,x) = x$ is __________ .

5. Let $A := \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{Q} \text{ and } y \in \mathbb{R} \setminus \mathbb{Q}\}$. The number of connected components of $\mathbb{R}^2 \setminus A$ is __________ .

3 Questions with one or more correct answers [66 marks]

1. The equation $4\sin^2 x + 10x^2 = \cos x$ has
   (a) no real solution.
   (b) exactly one real solution.
   (c) exactly two real solution.
   (d) more than two real solution.

2. Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function such that $f(0) = 0$, $f(\frac{1}{2}) = 5$, and $|f(z)| < 10$ for $|z| < 1$. Then
   (a) the set $\{z \in \mathbb{C} : |f(z)| = 5\}$ is unbounded.
   (b) the set $\{z \in \mathbb{C} : |f'(z)| = 5\}$ is a circle of positive radius.
   (c) $f(1) = 10$.
   (d) for all points $z \in \mathbb{C}$, $f''(z) = 0$.

3. Let A be a $n \times n$ matrix with complex entries such that $A^m = I$ for some positive integer m. Then
   (a) A is a diagonalisable matrix.
   (b) A is a similar to a triangular matrix but not A need not be a diagonalisable matrix.
   (c) all the eigen values of A are roots of unity.
   (d) none of the above.

4. Let G be a group of order 75. Then the group G
   (a) is cyclic.
   (b) has an element of order 25.
   (c) has an element of order 5.

(d) has an element of order 15.

5. Let $R := C([0,1], \mathbb{R})$ be the ring of all continuous real valued functions on $[0,1]$ and let $I := \{f \in R : f(1/2) = f(1/3) = 0\}$. Then

   (a) I is not an ideal in R.

   (b) I is an ideal of R but not a prime ideal in R.

   (c) I is a prime ideal but not a maximal ideal in R.

   (d) I is a maximal ideal in $\mathbb{R}$.

6. Let v be non-trivial solution to the equation $y(x) = x - \int_0^x (x-t)y(t)dt$. Then

   (a) $v(n\pi) = 0$ for all $n \in \mathbb{Z}$.

   (b) the function v has only finitely many zeros.

   (c) the function v is unbounded.

   (d) there exists a function $u : \mathbb{R} \to (-\infty, 0)$ such that $u(x) > v(x)$ for all $x \in \mathbb{R}$.

7. Let $(X, \| \,\|) = (\mathbb{R}^2, \| \,\|_\infty)$ and $Y := \{(x,y) \in \mathbb{R}^2 : x - 3y = 0\}$. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the linear functional defined by $f(x,y) = x + 3y$. Let $f_1, f_2 : Y \to \mathbb{R}$ be the linear functionals defined by $f_1(x,y) = x$ and $f_2(x,y) = 3y$. Then the linear functional f is Hahn-Banach extension of

   (a) f_1 but not f_2.

   (b) f_2 but not f_1.

   (c) both f_1 and f_2.

   (d) neither f_1 nor f_2.

8. Let $D : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be a compact operator, which is diagonal with respect to the standard orthonormal basis of $\ell^2(\mathbb{N})$. Then

   (a) 0 is necessarily an eigenvalue of D.

   (b) 0 need not be an eigenvalue of D.

   (c) there exists a sequence of eigenvalues of D converging to 0.

   (d) 0 need not be a limit point of eigenvalues of D.

9. Let u be a continuously differentiable function that satisfies

$$\frac{\partial u(t,x)}{\partial t} = \frac{\partial u(t,x)}{\partial x}, \quad (t,x) \in (0,\infty) \times (0,1),$$

$$u(0,x) = \begin{cases} 0, & x \le 1/4, \\ 1 - \exp\left(\frac{4e^{-2/(4x-1)}}{4x-3}\right), & x \in (1/4, 3/4), \\ 1 & x \ge 3/4, \end{cases}$$

$$u(t,0) = 0, \quad t \in (0,\infty).$$

Then the function

(a) u is not well defined.
(b) u is well defined and $u(1/4,1) = 0$.
(c) u is well defined and $u(1/4,1) = 1$.
(d) u is well defined and $u(1,1) = 0$.

10. Let $\Omega = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ with its boundary $\partial\Omega$ and $\mathcal{I}_n v : \partial\Omega \to \mathbb{R}$, given by

$$(\mathcal{I}_n v)(\cos\theta, \sin\theta) = \sum_{j=0}^{n} v(\cos(2\pi j/n), \sin(2\pi j/n)) \prod_{\substack{k=0 \\ k \ne j}}^{n} \frac{\theta - \theta_k}{\theta_j - \theta_k}, \quad \theta \in [0, 2\pi],$$

where $\theta_j = 2\pi j/n, j = 0, \ldots, n$, is the Lagrange interpolant of a smooth function v at $n+1$ equidistant interpolation points θ_j. If u_n is the solution of the following boundary value problem

$$\begin{aligned} \Delta u &= 0, && \text{in } \Omega, \\ u &= \mathcal{I}_n v, && \text{on } \partial\Omega, \end{aligned}$$

for $v(x,y) = x^2 + y^2$ then, $\|u_n - v\|_{\infty,\overline{\Omega}} = \max_{(x,y)\in\overline{\Omega}} |u_n(x,y) - v(x,y)|$ satisfies

(a) $||u_n - v||_{\infty,\overline{\Omega}} = 0$.
(b) $||u_n - v||_{\infty,\overline{\Omega}} = 1$.
(c) $\lim_{n\to\infty} \|u_n - v\|_{\infty,\overline{\Omega}} = 0$.
(d) $\lim_{n\to\infty} \|u_n - v\|_{\infty,\overline{\Omega}} = 1$.

11. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by $f(x,y) := (x^2 + y^2)e^{-x^2-y^2}$. Then

(a) the point $(0,0)$ a global minimum for the function f.
(b) the function f does not have a maximum.
(c) the function f attains its maximum at a point in $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
(d) the function f has a saddle point in $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \ne 1\}$.