Grade Table (for checker use only)

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

Team Members:

- ....
- ....
- ....

INSTRUCTIONS:

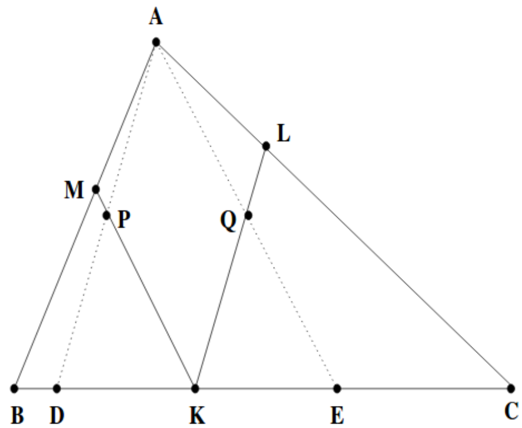
- **Write your team name on top of each page.**
- If you have any queries, contact an invigilator. Any sort of interaction with another team can lead to a penalty or disqualification.
- Submit any electronic devices that you possess, to one of the invigilators. You may collect them after the event. Any team caught using any electronic device will be immediately disqualified.
- Enough space has been provided in the question paper. Use it wisely. However, if you need extra sheets, contact an invigilator.

1. (10 points) Three gods A, B, and C are called, in no particular order, $True$, $False$, and *Random*. $True$ always speaks truly, $False$ always speaks falsely, but whether *Random* speaks truly or falsely is a completely random matter. Your task is to determine the identities of A, B, and C by asking three yes-no questions; each question must be put to exactly one god. The gods understand English, but will answer all questions in their own language, in which the words for *yes* and *no* are *da* and ja, in some order. You do not know which word means which.
Solution:
Q1: Ask god B, "If I asked you 'Is A *Random*?', would you say ja?". If B answers ja, either B is *Random* (and is answering randomly), or B is not *Random* and the answer indicates that A is indeed *Random*. Either way, C is not *Random*. If B answers *da*, either B is *Random* (and is answering randomly), or B is not *Random* and the answer indicates that A is not *Random*. Either way, you know the identity of a god who is not *Random*.
Q2: Go to the god who was identified as not being *Random* by the previous question (either A or C), and ask him: "If I asked you 'Are you $False$?', would you say ja?". Since he is not *Random*, an answer of *da* indicates that he is $True$ and an answer of ja indicates that he is $False$. This question can also be simplified: "Does '*da*' mean '*yes*'?"
Q3: Ask the same god the question: "If I asked you 'Is B *Random*?', would you say ja?". If the answer is ja, B is *Random*; if the answer is *da*, the god you have not yet spoken to is *Random*. The remaining god can be identified by elimination.

2. (10 points) The in-circle of triangle ABC touches the sides BC, CA and AB in K, L and M respectively. The line through A and parallel to LK meets MK in P and the line through A and parallel to MK meets LK in Q. Show that the line PQ bisects the sides AB and AC of triangle ABC.

   Solution:

   Let AP, AQ produced meet BC in D, E respectively.



Since MK is parallel to AE, we have $\angle AEK = \angle MKB$. SInce $BK = BM$, both being tangents to the circle from B, $\angle MKB = \angle BMK$. This with the fact that MK is parallel to AE gives us $\angle AEK = \angle MAE$. This shows that $MAEK$ is an isosceles trapezoid. We conclude that $MA = KE$. Similarly, we can prove that $AL = DK$. But $AM = AL$. We get $DK = KE$. Since KP is parallel to AE, we get $DP = PA$ and similarly $EQ = QA$. This implies that PQ is parallel to DE and hence bisects AB, AC when produced.

[The same argument even if one or both of P and Q lie outside triangle ABC.]

3. (10 points) Let $a_1, a_2, ..., a_n$ be arbitrary real numbers. Show that the following will always hold

$$\frac{a_1}{1+a_1^2} + \frac{a_2}{1+a_1^2+a_2^2} + ... + \frac{a_n}{1+a_1^2+a_2^2+...+a_n^2} < \sqrt{n}$$

Solution:
By the arithmetic-quadratic mean inequality, it suffices to prove that

$$\frac{a_1^2}{(1+a_1^2)^2} + \frac{a_2^2}{(1+a_1^2+a_2^2)^2} + \cdots + \frac{a_n^2}{(1+a_1^2+a_2^2+\cdots+a_n^2)^2} < 1.$$

Observe that for $k \geq 2$ the following holds :

$$\frac{a_k^2}{(1+a_1^2+\cdots+a_n^2)^2} \leq \frac{a_k^2}{(1+\cdots+a_{k-1}^2)(1+\cdots+a_k^2)} = \frac{1}{1+a_1^2+\cdots+a_{k-1}^2} - \frac{1}{1+a_1^2+\cdots+a_k^2}.$$

For $k=1$ we have $\frac{a_1^2}{(1+a^1)^2} \leq 1 - \frac{1}{1+a_1^2}$. Summing these inequalities, we obtain

$$\frac{a_1^2}{(1+a_1^2)^2} + \cdots + \frac{a_n^2}{(1+a_1^2+\cdots+a_n^2)^2} \leq 1 - \frac{1}{1+a_1^2+\cdots+a_n^2} < 1.$$

4. (10 points) We have 2^m sheets of paper, with the number 1 written on each of them. We per form the following operation. In every step we choose two distinct sheets; if the numbers on the two sheets are a and b, then we erase these numbers and write the number $a+b$ on both sheets. Prove that after $m2^{m-1}$ steps, the sum of the numbers on all the sheets is at least 4^m.

   Solution:

   Let P_k be the product of the numbers on the sheets after k steps. Suppose that in the $(k+1)^{th}$ step the numbers a and b are replaced by $a+b$. In the product, the number ab is replaced by $(a+b)^2$, and the other factors do not change. Since $(a+b)^2 \geq 4ab$, we see that $P_{k+1} \geq 4P_k$. Starting with $P_0 = 1$, a straightforward induction yields

   $$P_k \geq 4^k$$

   for all integers $k \geq 0$; in particular

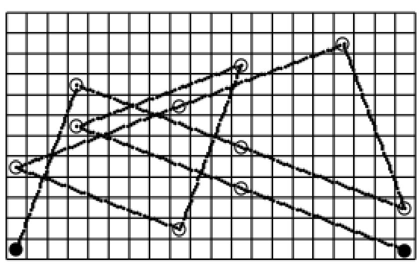
   $$P_{m \cdot 2^{m-1}} \geq 4^{m \cdot 2^{m-1}} = (2^m)^{2^m},$$

   so by the AM–GM inequality, the sum of the numbers written on the sheets after $m2^{m-1}$ steps is at least

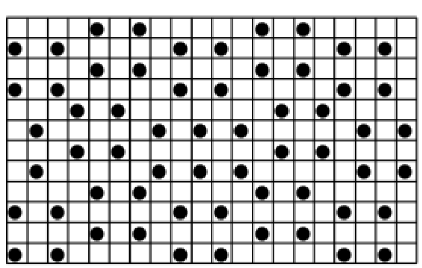
   $$2^m \cdot \sqrt[2^m]{P_{m \cdot 2^{m-1}}} \geq 2^m \cdot 2^m = 4^m.$$

5. (10 points) We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20$, $|BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.
   (a) Show that the task cannot be done if r is divisible by 2 or 3.
   (b) Prove that the task is possible when $r = 73$.
   (c) Can the task be done when $r = 97$?

Solution:
We shall work on the array of lattice points defined by $\mathcal{A} = (x, y) \in \mathbb{Z}^2 | 0 \le x \le 19, 0 \le y \le 11$. Our task is to move from $(0,0)$ to $(19,0)$ via the points of $\mathcal{A}$ so that each move has the form $(x,y) \to (x+a, y+b)$, where $a, b \in \mathbb{Z}$ and $a^2 + b^2 = r$.
(a) If r is even, then $a + b$ is even whenever $a^2 + b^2 = r (a, b \in \mathbb{Z})$. Thus the parity of $x + y$ does not change after each move, so we cannot reach $(19,0)$ from $(0,0)$.
If $3|r$, then both a and b are divisible by 3, so if a point (x,y) can be reached from $(0,0)$, we must have $3|x$. Since $3 \nmid 19$, we cannot get to $(19,0)$.
(b) We have $r = 73 = 8^2 + 3^2$, so each move is either $(x,y) \to (x \pm 8, y \pm 3)$ or $(x,y) \to (x \pm 3, y \pm 8)$. One possible solution is



(c) We have $97 = 9^2 + 4^2$. Let us partition $\mathcal{A}$ as $\mathcal{B} \cup \mathcal{C}$, where $\mathcal{B} = (x, y) \in A | 4 \le y \le 7$. It is easily seen that moves of the type $(x,y) \to (x \pm 9, y \pm 4)$ always take us from the set $\mathcal{B}$ to $\mathcal{C}$ and vice versa, while the moves $(x,y) \to (x \pm 4, y \pm 9)$ always take us from $\mathcal{C}$ to $\mathcal{C}$. Furthermore, each move of the type $(x,y) \to (x \pm 9, y \pm 4)$ changes the parity of x, so to get from $(0,0)$ to $(19,0)$ we must have an odd number of such moves. On the other hand, with an odd number of such moves, starting from $\mathcal{C}$ we can end up only in $\mathcal{B}$, although the point $(19,0)$ is not in B. Hence, the answer is *no*.



6. (10 points) Let k be a positive integer and m be an odd number. Prove that there exists a natural number n such that $n^n - m$ is divisible by 2^k.

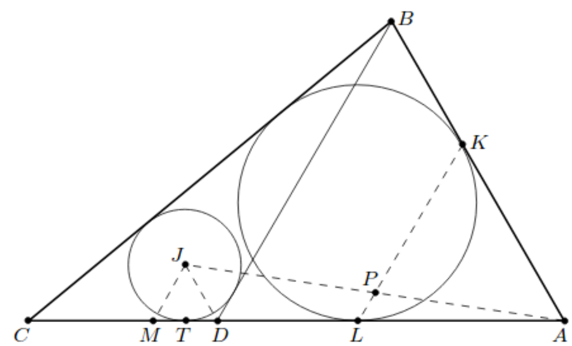
   Solution:

   Proof by induction on k.
   Base case : $k = 1$
   $n = 1$ works.
   Induction hypothesis : Suppose there exists n_0 such that $2^k | n_0^{n_0} - m$.
   Since m is odd, n_0 is also odd.
   If $n_0^{n_0} \equiv m \pmod 2)^{k+1}$, then $n = n_0$ works.
   Otherwise $n_0^{n_0} = m + s2^k$, where s is odd.
   Let $s = 2l + 1$.
   $n_0^{n_0} \equiv m + 2^k \pmod 2)^{k+1}$.
   Choose $n = n_0 + 2^k$.
   Claim : $n^n \equiv m \pmod 2)^{k+1}$
   $\phi(2^{k+1}) = 2^k$ and $gcd(n, 2^{k+1}) = 1$
   $\implies n^{2^k} \equiv 1 \pmod{2^{k+1}}$
   $\implies n^n = n^{n_0+2^k} = n^{n_0} \cdot n^{2^k} \equiv n^{n_0} \pmod{2^{t+1}}$.
   $\implies n^{n_0} = (n + 2^k)_{n_0} = {n_0}^{n_0} + n_0 \cdot {n_0}^{n_0 - 1} + 2^{2k}(\cdots)$
   $\implies n^n \equiv {n_0}^{n_0}(1 + 2^k) \pmod{2^{k+1}}$
   $\implies n^n \equiv (m + 2^k)^(1 + 2^t) \pmod{2^{t+1}}$
   $\implies n^n \equiv m + 2^t(1 + m) \pmod{2^{t+1}}$
   $\implies n^n \equiv m \pmod{2^{t+1}}$.

7. (10 points) A point D is chosen on side AC of triangle ABC with $\angle C < \angle A < 90^o$ in such a way that $BD = BA$. The incircle of triangle ABC is tangent to AB and AC at points K and L, respectively. Let J be the incentre of triange BCD. Prove that the line KL bicects line segment AJ.

   Solution:

   Denote by P the common point of AJ and KL. Let the parallel to KL through J meet AC at M. Then P is the midpoint of AJ if and only if $AM = 2 \cdot AL$, which we are about to show.



   Denoting $\angle BAC = 2\alpha$, the equalities $BA = BD$ and $AK = AL$ imply $\angle ADB = 2\alpha$ and $\angle ALK = 90^o - \alpha$. Since DJ bisects $\angle BDC$, we obtain $\angle CDJ = 90^o - \alpha$. Also $\angle DMJ = \angle AKL = 90^o - \alpha$ since $JM||KL$. It follows that $JD = JM$.

   Let the incircle of triangle BCD touch its side CD at T. Then $JT \perp CD$, meaning that JT is the altitude to base DM of the isosceles triangle DMJ. It now follows that $DT = MT$, and we have

$$DM = 2 \cdot DT = BD + CD - BC.$$

   Therefore

$$\begin{aligned} AM &= AD + (BD + CD - BC) \\ &= AD + AB + CD - BC \\ &= AC + AB - BC \\ &= 2 \cdot AL, \end{aligned}$$

   which completes the proof.

8. (10 points) Let $n > 1$ be a given positive integer. Prove that infinitely many terms of the sequence $(a_k)_{k\geq 1}$, defined by

$$a_k = \left\lfloor \frac{n^k}{k} \right\rfloor,$$

are odd.

Solution:

If n is odd, let $k = n^m$ for $m = 1, 2, \dots$. Then $a^k = n^{n^m - m}$, which is odd for each m. Henceforth, assume that n is even, say $n = 2t$ for some integer $t \geq 1$. Then, for any $m \geq 2$, the integer $n^{2^m} - 2^m = 2^m(2^{2^m - m} \cdot t^{2^m} - 1)$ has an odd prime divisor p, since $2^m - m > 1$. Then, for $k = p.2^m$, we have

$$n^k = (n^{2^m})^p \equiv (2^m)^p = (2^p)^m \equiv 2^m,$$

where the congruences are taken modulo p.

Also, from $n^k - 2^m < n^k < n^k + 2^m(p-1)$, we see that the fraction $\frac{n^k}{k}$ lies strictly between the consecutive integers $\frac{n^k - 2^m}{p \cdot 2^m}$ and $\frac{n^k + 2^m(p-1)}{2^m}$, which gives

$$\left\lfloor \frac{n^k}{k} \right\rfloor = \frac{n^k - 2^m}{p \cdot 2^m}.$$

We finally observe that $\frac{n^k - 2^m}{p \cdot 2^m} = \frac{\frac{n^k}{2^m} - 1}{p}$ is an odd integer, since the integer $\frac{n^k}{2^m} - 1$ is odd (recall that $k > m$). Note that for different values of m, we get different values of k, due to the different powers of 2 in the prime factorisation of k.