# CFD in Engineering Applications

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- CAE Background
- Basics
- CFD – what, where and why ?
- CFD - fundamentals
- CFD – essentials
- Examples

# Introduction

- What is Computational Fluid Dynamics(CFD)?
- Why and where use CFD?
- Physics of Fluid
- Navier-Stokes Equation
- Numerical Discretization
- Grids
- Boundary Conditions
- Numerical Staff

# Role of CAE (CFD)

- **Computer-Aided Engineering** (CAE) is the broad usage of computer software to aid in engineering analysis tasks.

  - ✓ Finite Element Analysis (FEA)
  - ✓ Computational Fluid Dynamics (CFD)
  - ✓ Multi-body dynamics (MBD)
  - ✓ Optimization

- Software tools that have been developed to support these activities are considered CAE tools.

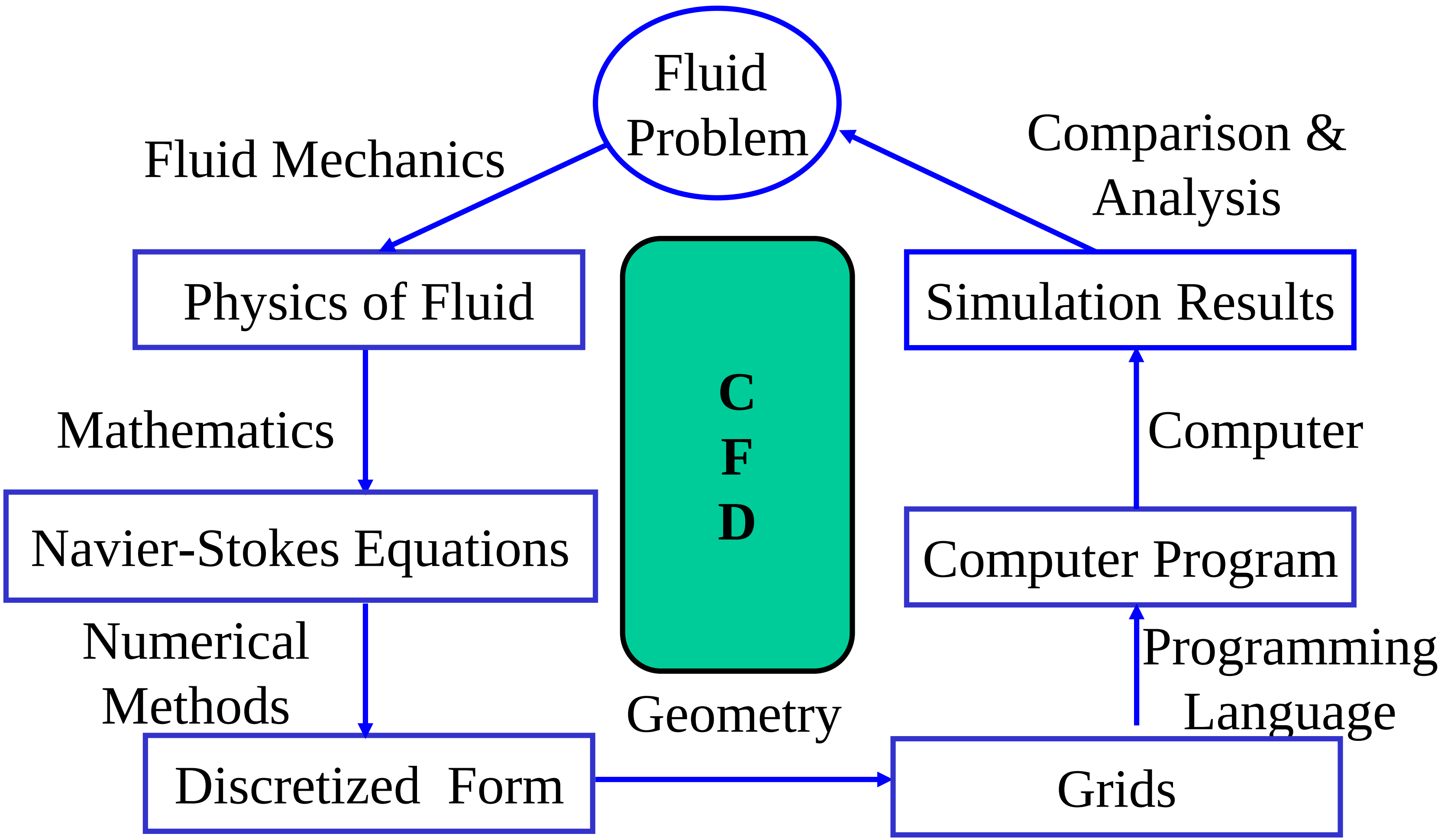
- The term encompasses simulation, validation, and optimization of products and manufacturing tools.

*In the future, CAE systems will be major providers of information to help support design teams in decision making !!*

# What is CFD?

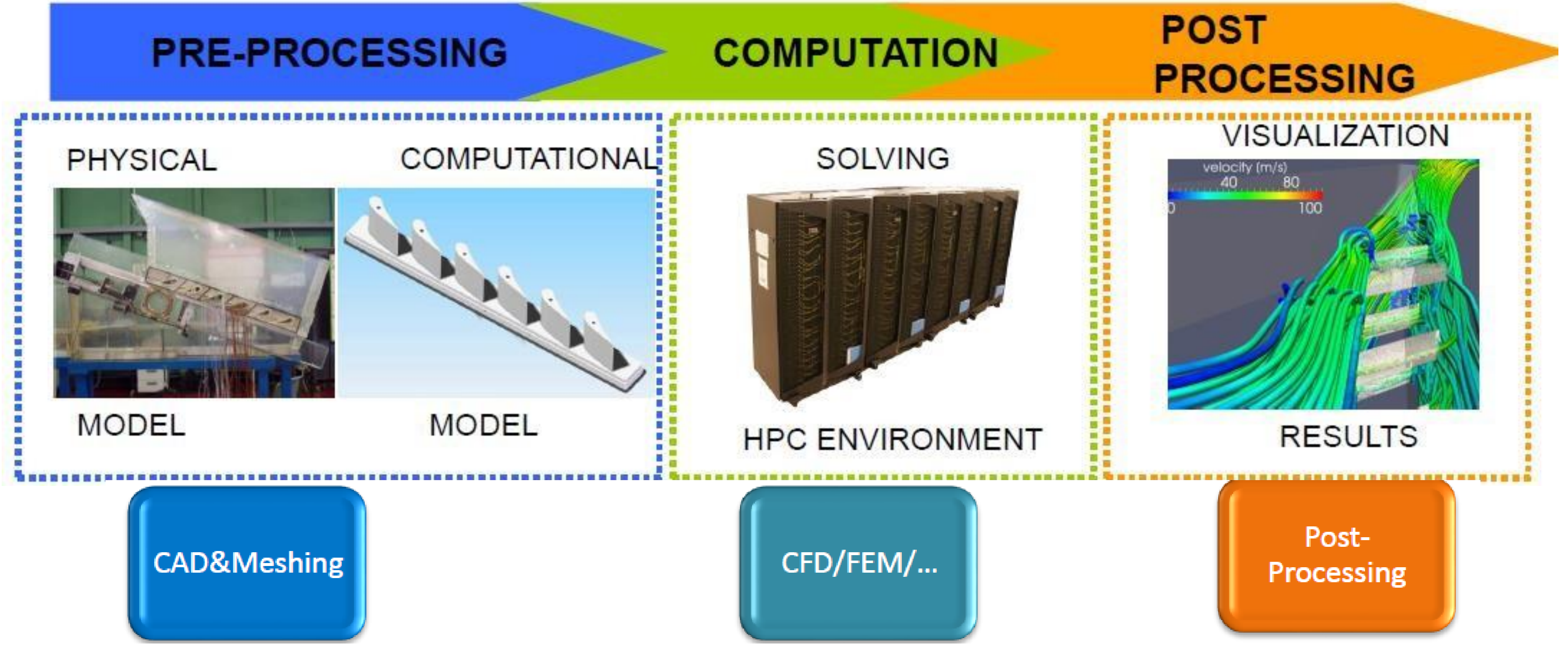
- Computational fluid dynamics (CFD) deals with solution of fluid dynamics and heat transfer problems using numerical techniques.
- CFD is an alternative to measurements for solving large-scale fluid dynamical systems.
- CFD has evolved as a design tool for various industries namely Aerospace, Mechanical, Auto-mobile, Chemical, Metallurgical, Electronics, and even Food processing industries.
- CFD is becoming a key-element for computer-aided designs in industries across world over.

# What is CFD?


Fluid Problem
Fluid Mechanics
Comparison & Analysis
Physics of Fluid
C F D
Simulation Results
Mathematics
Computer
Navier-Stokes Equations
Computer Program
Numerical Methods
Programming Language
Geometry
Discretized Form
Grids

# Phase of CFD

- **Pre-processing** – defining the geometry model, the physical model and the boundary conditions
- **Computing** (usually performed on high powered computers (HPC))
- **Post-processing of results** (using scientific visualization tools & techniques )




Iterative process !!

# CFD

- **CFD is the “science” of predicting fluid behaviour**
  - **Flow field, heat transfer, mass transfer, chemical reactions, etc…**
    - By solving the governing equations of fluid flow using a numerical approach (computer based simulation)
- **The results of CFD analyses**
  - **Represent valid engineering data that may be used for**
    - Conceptual studies of new designs (with reduction of lead time and costs)
    - Studies where controlled experiments are difficult to perform
    - Studies with hazardous operating conditions
    - Redesign engineering
- **CFD analyses represent a valid**
  - **Complement to experimental tests**
    - Reducing the total effort required in laboratory tests

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# Why use CFD?

- Analysis and Design
  - Simulation-based design instead of "build & test"
    - ✓ More cost effectively and more rapidly than with experiments
    - ✓ CFD solution provides high-fidelity database for interrogation of flow field
  - Simulation of physical fluid phenomena that are difficult to be measured by experiments
    - ✓ Scale simulations (e.g., full-scale ships, airplanes)
    - ✓ Hazards (e.g., explosions, radiation, pollution)
    - ✓ Physics (e.g., weather prediction, planetary boundary layer, stellar evolution)
- Knowledge and exploration of flow physics

# Why use CFD?

| | Simulation(CFD) | Experiment |
|---|---|---|
| Cost | Cheap | Expensive |
| Time | Short | Long |
| Scale | Any | Small/Middle |
| Information | All | Measured Points |
| Repeatable | All | Some |
| Security | Safe | Some Dangerous |

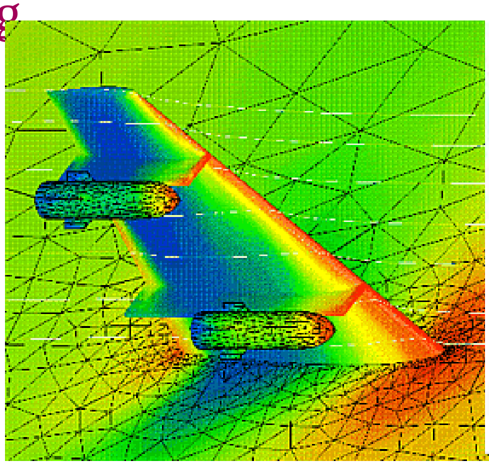
# Why use CFD?

- Computers built in the 1950s performed limited floating point operations per second, i.e. only few hundred arithmetic operations per second.
- Computers that are manufactured today have *teraflops* rating where *tera* is a trillion and *flops* is an abbreviation for floating point operations per second.
- While computer speed has increased at a tremendous rate, computer cost has fallen significantly.
- It is revealed that the computational cost has been reduced by approximately a factor of 10 every 8 years.
- Today a desktop machine can do the job of "mainframe" machines of 1980s.

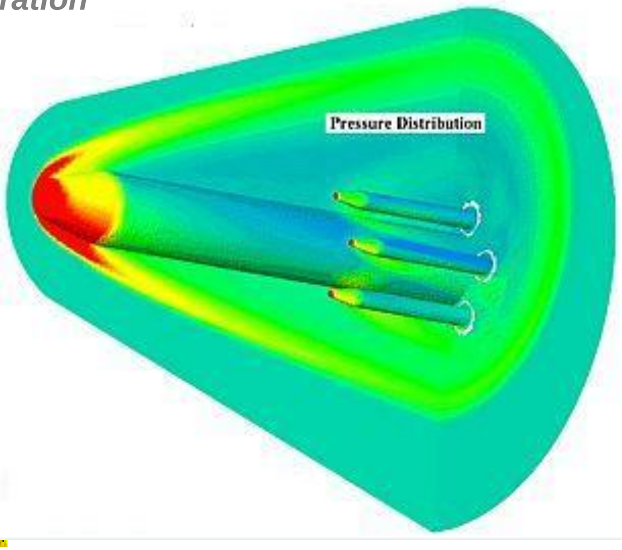
# Where is CFD used? (Aerospace)

- Where is CFD used?
  - ***Aerospace***
  - Appliances
  - Automotive
  - Biomedical
  - Chemical Processing
  - HVAC&R
  - Hydraulics
  - Marine
  - Oil & Gas
  - Power Generation
  - Sports

*F18 Store Separation*



*Wing-Body Interaction*

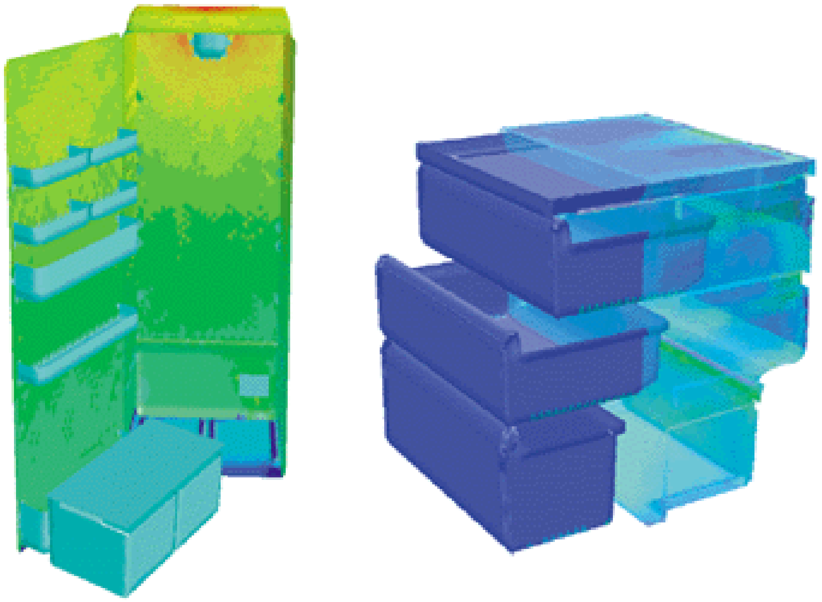



*Hypersonic Launch Vehicle*

Source: internet

# Where is CFD used? (Appliances)

- Where is CFD used?
  - Aerospace
  - ***Appliances***
  - Automotive
  - Biomedical
  - Chemical Processing
  - HVAC&R
  - Hydraulics
  - Marine
  - Oil & Gas
  - Power Generation
  - Sports

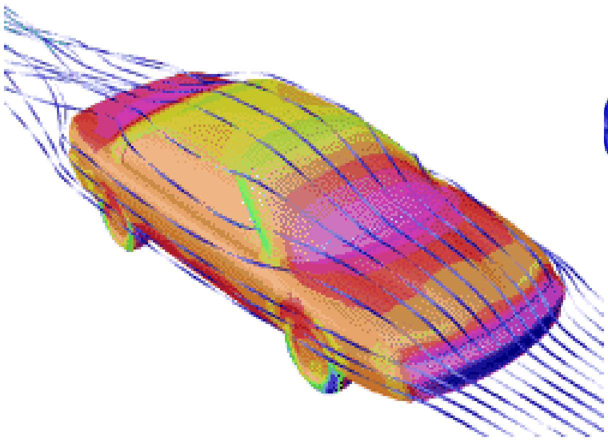


***Surface-heat-flux plots of the No-Frost refrigerator and freezer compartments helped BOSCH-SIEMENS engineers to optimize the location of air inlets.***

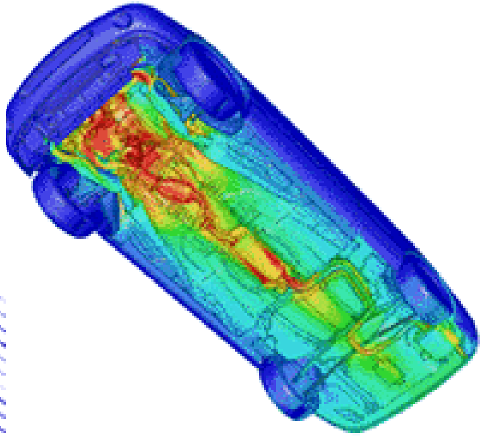
Source: internet

# Where is CFD used? (Automotive)

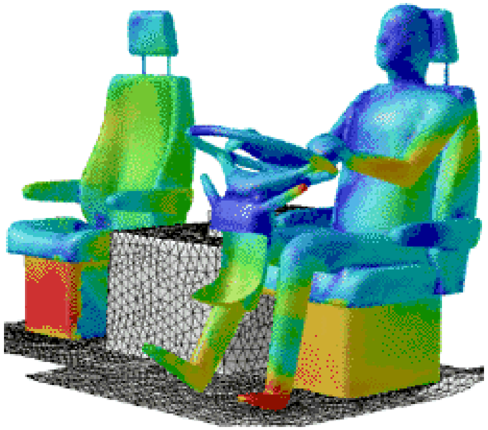
- Where is CFD used?
  - Aerospace
  - Appliances
  - ***Automotive***
  - Biomedical
  - Chemical Processing
  - HVAC&R
  - Hydraulics
  - Marine
  - Oil & Gas
  - Power Generation
  - Sports



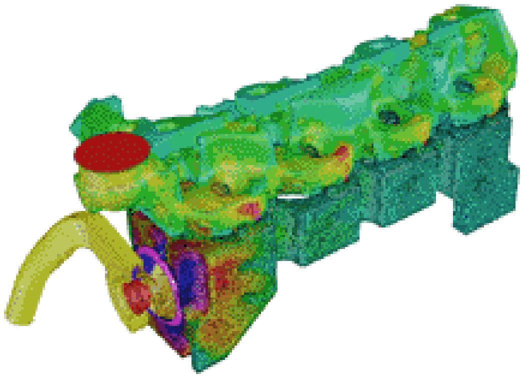
***External Aerodynamics***



***Undercarriage Aerodynamics***



***Interior Ventilation***

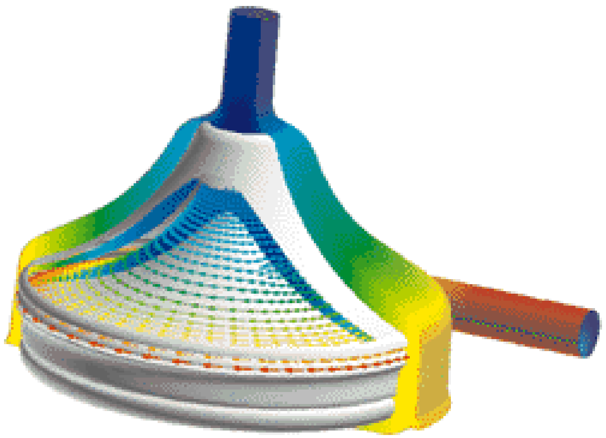


***Engine Cooling***

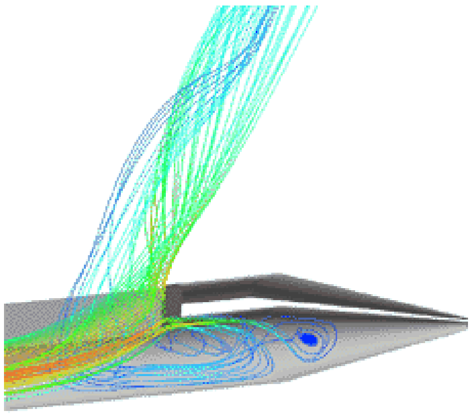
Source: internet

# Where is CFD used? (Biomedical)

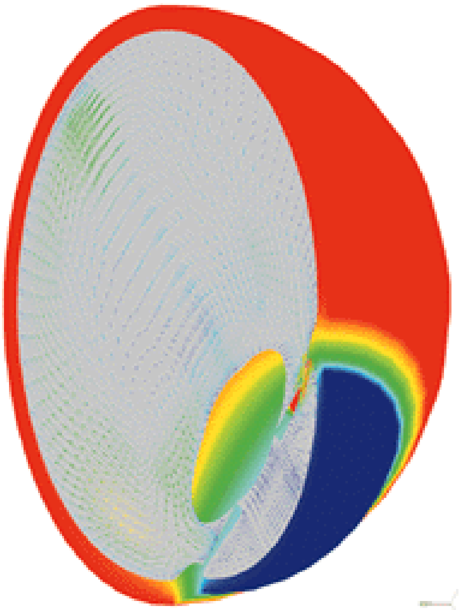
- Where is CFD used?
  - Aerospace
  - Appliances
  - Automotive
  - ***Biomedical***
  - Chemical Processing
  - HVAC&R
  - Hydraulics
  - Marine
  - Oil & Gas
  - Power Generation
  - Sports



***Medtronic Blood Pump***



***Spinal Catheter***

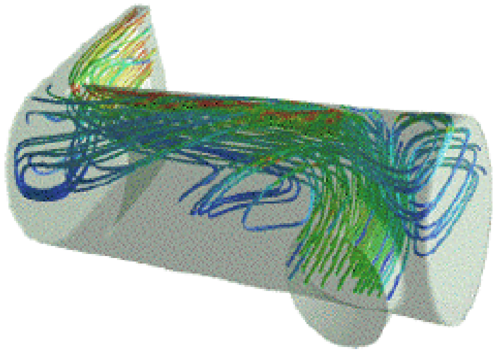


***Temperature and natural convection currents in the eye following laser heating.***

Source: internet

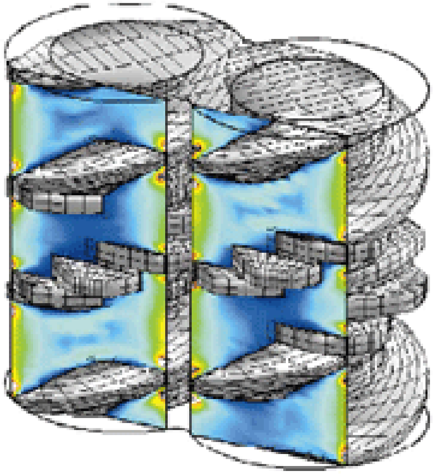
# Where is CFD used? (Chemical Processing)

- Where is CFD used?
  - Aerospace
  - Appliances
  - Automotive
  - Biomedical
  - ***Chemical Processing***
  - HVAC&R
  - Hydraulics
  - Marine
  - Oil & Gas
  - Power Generation
  - Sports



***Polymerization reactor vessel - prediction of flow separation and residence time effects.***

***Twin-screw extruder modeling***

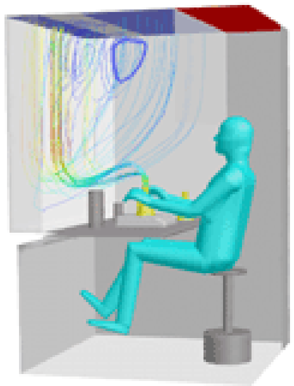


***Shear rate distribution in twin-screw extruder simulation***

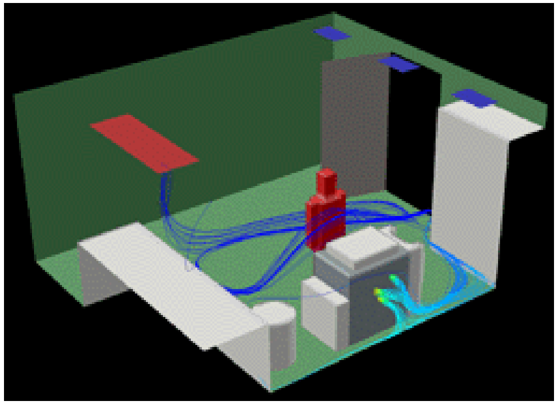
Source: internet

# Where is CFD used? (HVAC&R)

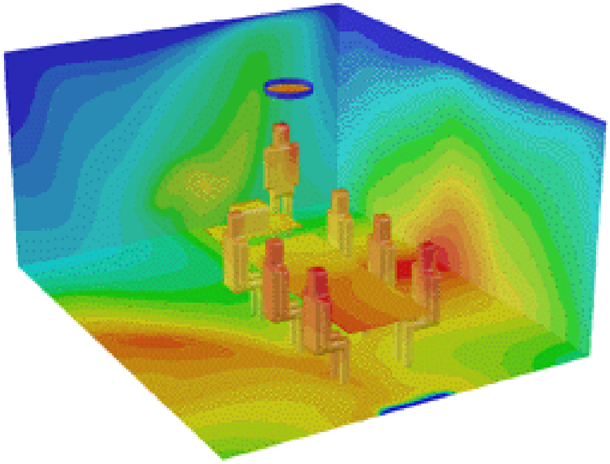
- Where is CFD used?
  - Aerospace
  - Appliances
  - Automotive
  - Biomedical
  - Chemical Processing
  - ***HVAC&R***
  - Hydraulics
  - Marine
  - Oil & Gas
  - Power Generation
  - Sports



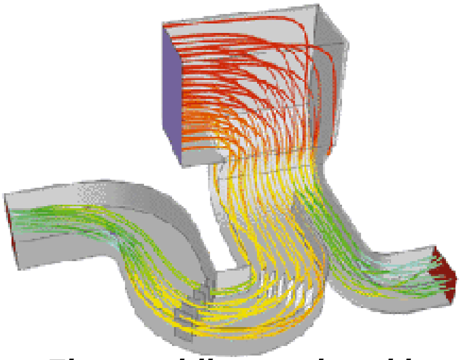
***Streamlines for workstation ventilation***



***Particle traces of copier VOC emissions colored by concentration level fall behind the copier and then circulate through the room before exiting the exhaust.***



***Mean age of air contours indicate location of fresh supply air***



***Flow pathlines colored by pressure quantify head loss in ductwork***

Source: internet

# Where is CFD used? (Hydraulics)

- Where is CFD used?
  - Aerospace
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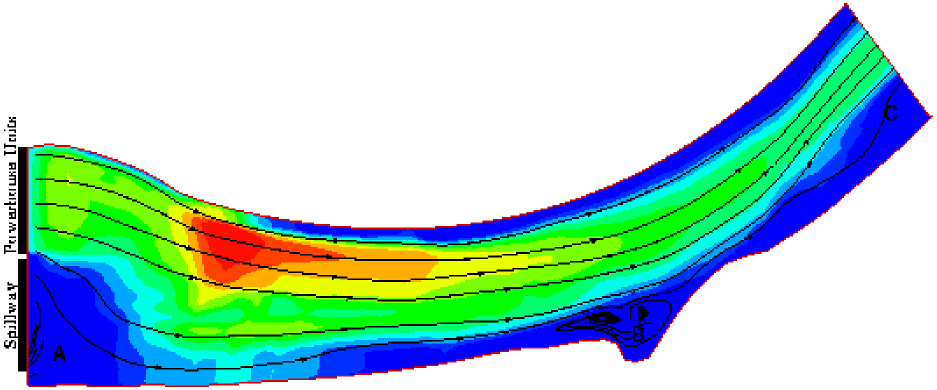
  - Automotive
  - Biomedical
  - Chemical Processing
  - HVAC&R
  - ***Hydraulics***
  - Marine
  - Oil & Gas
  - Power Generation
  - Sports

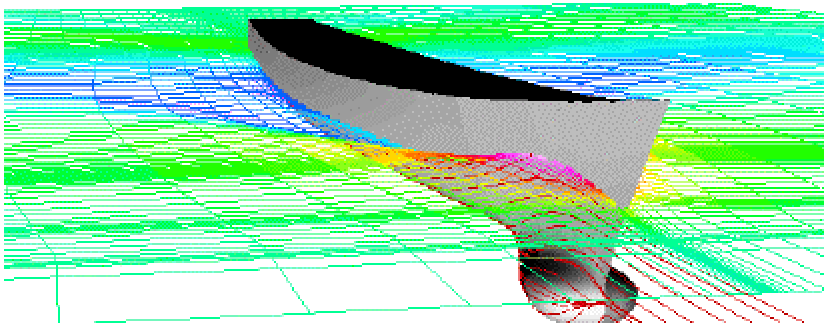



Total Discharge = 125,000 cfs (no flow through spillway)

Source: internet

# Where is CFD used? (Marine)

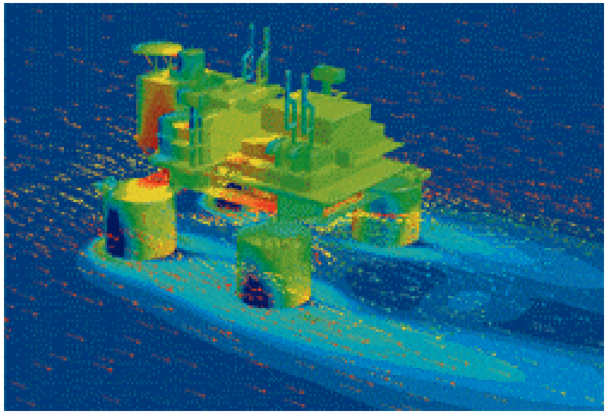
- Where is CFD used?
  - Aerospace
  - Appliances
  - Automotive
  - Biomedical
  - Chemical Processing
  - HVAC&R
  - Hydraulics
  - ***Marine***
  - Oil & Gas
  - Power Generation
  - Sports



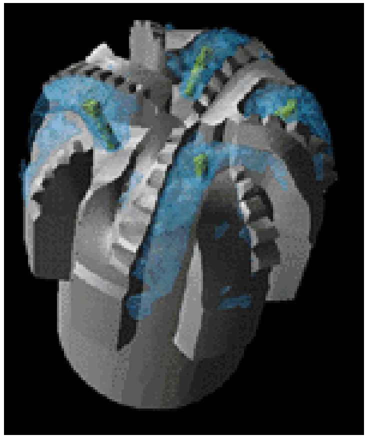
Source: internet

# Where is CFD used? (Oil & Gas)

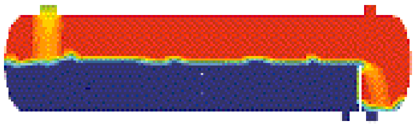
- Where is CFD used?
  - Aerospace
  - Appliances
  - Automotive
  - Biomedical
  - Chemical Processing
  - HVAC&R
  - Hydraulics
  - Marine
  - ***Oil & Gas***
  - Power Generation
  - Sports



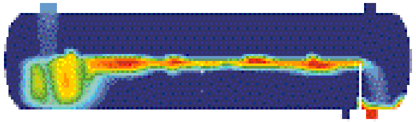
***Flow vectors and pressure distribution on an offshore oil rig***



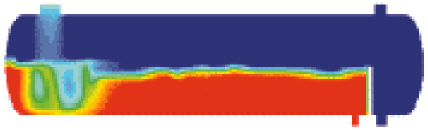
***Flow of lubricating mud over drill bit***



***Volume fraction of gas***



***Volume fraction of oil***



***Volume fraction of water***

***Analysis of multiphase separator***

Source: internet

# Where is CFD used? (Power Generation)

- Where is CFD used?
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  - Hydraulics
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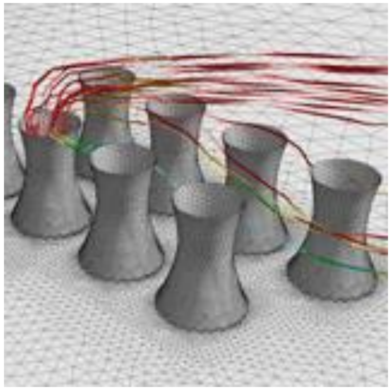
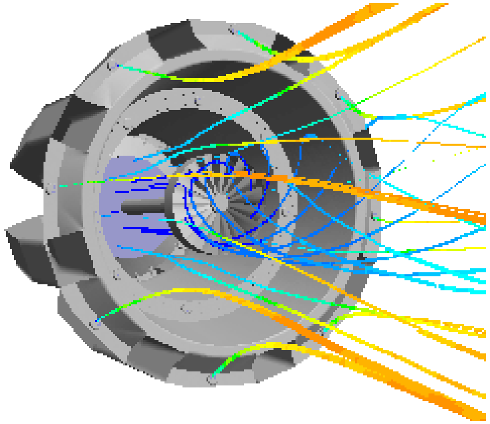
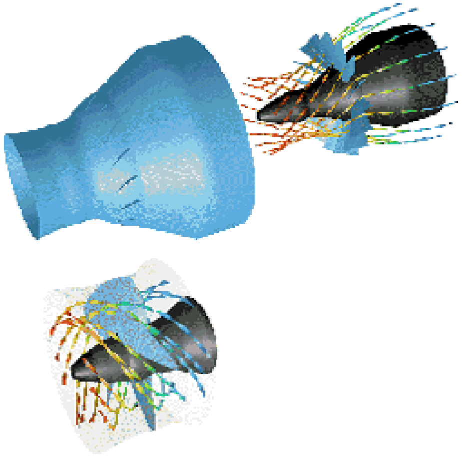


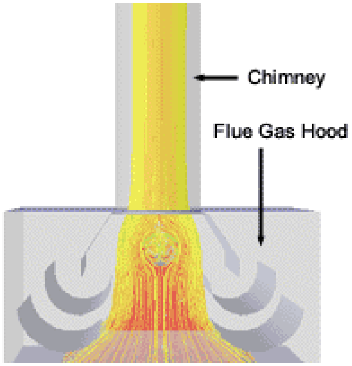
***Flow around cooling towers***



***Flow in a burner***



***Flow pattern through a water turbine.***




***Pathlines from the inlet colored by temperature during standard operating conditions***

Source: internet

# Where is CFD used? (Sports)

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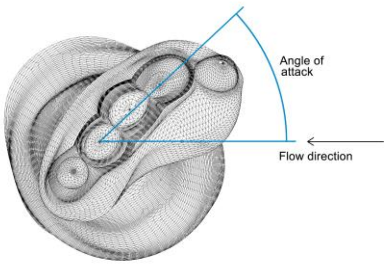



Source: internet

# Introduction

- What is Computational Fluid Dynamics(CFD)?
- Why and where use CFD?
- Physics of Fluid
- Navier-Stokes Equation
- Numerical Discretization
- Grids
- Boundary Conditions
- Numerical Staff

# Physics of Fluid

- Fluid = Liquid + Gas
- Density $\rho$

$$\rho = \begin{cases} \text{const} & \text{incompressible} \\ \text{variable} & \text{compressible} \end{cases}$$

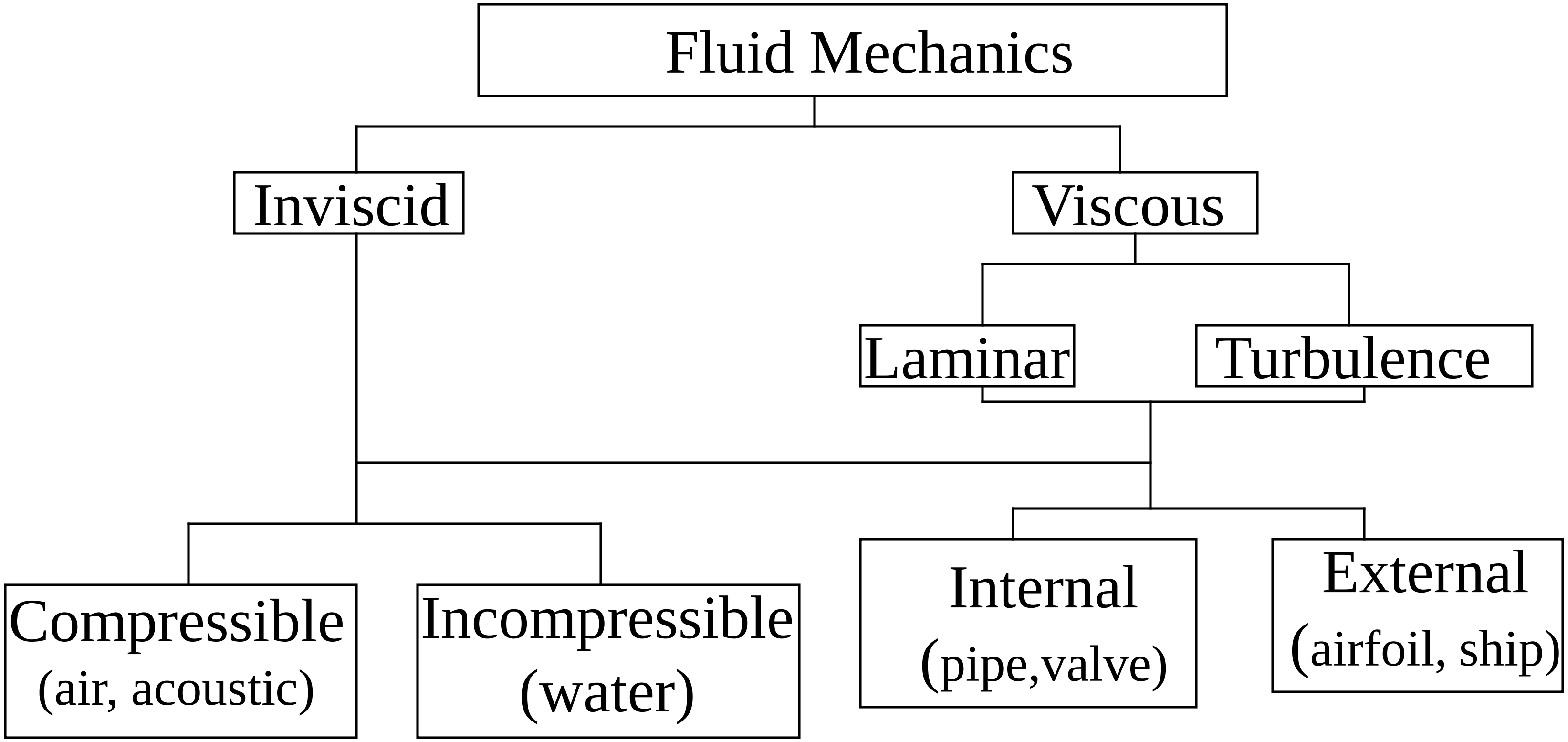
- Viscosity $\mu$:

***resistance*** to flow of a fluid

$$\mu = \left(\frac{Ns}{m^3}\right) = (Poise)$$

| Substance | Air(18ºC) | Water(20ºC) | Honey(20ºC) |
|---|---|---|---|
| Density(kg/m$^3$) | 1.275 | 1000 | 1446 |
| Viscosity(P) | 1.82e-4 | 1.002e-2 | 190 |

# Physics of Fluid




Components of Fluid Mechanics

# Physics of Fluid

- CFD codes typically designed for representation of specific flow phenomenon

  - Viscous vs. inviscid (no viscous forces) (Re)
  - Turbulent vs. laminar (Re)
  - Incompressible vs. compressible (Ma)
  - Single- vs. multi-phase (Ca)
  - Thermal/density effects and energy equation (Pr, $\gamma$, Gr, Ec)
  - Free-surface flow and surface tension (Fr, We)
  - Chemical reactions, mass transfer
  - etc…

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# Navier-Stokes Equations

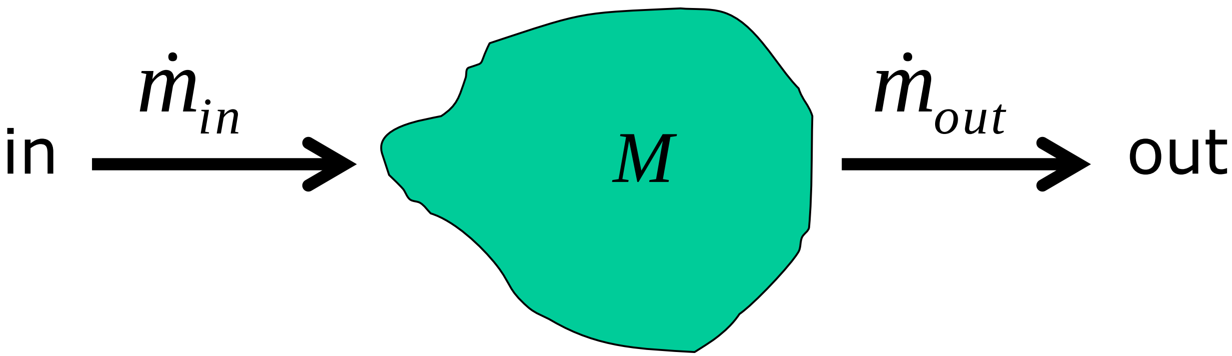
Claude-Louis Navier

George Gabriel Stokes

C.L. M. H. Navier, Memoire sur les Lois du Mouvements des Fluides, *Mem. de l'Acad. d. Sci.,* 6, 398 (1822)
C.G. Stokes, On the Theories of the Internal Friction of Fluids in Motion, Trans. Cambridge Phys. Soc., 8, (1845)

Source: internet

# Conservation law




$$\frac{dM}{dt} = \dot{m}_{in} - \dot{m}_{out}$$

$$\dot{m}_{in} = \dot{m}_{out}$$

$$\frac{dM}{dt} = 0$$

Mass
Momentum
Energy

# Navier-Stokes Equation I

- Mass Conservation→Continuity Equation

$$\frac{D\rho}{Dt} + \rho \frac{\partial U_i}{\partial x_i} = 0$$

Compressible

$$\rho = const, \frac{D\rho}{Dt} = 0$$

$$\frac{\partial U_i}{\partial x_i} = 0$$

Incompressible

# Navier-Stokes Equation II

- Momentum Conservation → Momentum Equation

$$\underbrace{\rho\frac{\partial U_j}{\partial t}}_{I}+\underbrace{\rho U_i\frac{\partial U_j}{\partial x_i}}_{II}=-\underbrace{\frac{\partial P}{\partial x_j}}_{III}-\underbrace{\frac{\partial \tau_{ij}}{\partial x_i}}_{IV}+\underbrace{\rho g_j}_{V}$$

$$\tau_{ij}=-\mu\left(\frac{\partial U_j}{\partial x_i}+\frac{\partial U_i}{\partial x_j}\right)+\frac{2}{3}\delta_{ij}\mu\frac{\partial U_k}{\partial x_k}$$

I : Local change with time

II : Momentum convection

III: Surface force

IV: Molecular-dependent momentum exchange(diffusion)

V: Mass force

# Navier-Stokes Equation III

- Momentum Equation for Incompressible Fluid

$$\frac{\partial \tau_{ij}}{\partial x_i} = -\mu \frac{\partial}{\partial x_i}\left(\frac{\partial U_j}{\partial x_i} + \frac{\partial U_i}{\partial x_j}\right) + \frac{2}{3}\delta_{ij}\mu \frac{\partial}{\partial x_i}\frac{\partial U_k}{\partial x_k}$$

$$\frac{\partial U_i}{\partial x_i} = 0$$

$$\longrightarrow \quad \frac{\partial \tau_{ij}}{\partial x_i} = -\mu \frac{\partial^2 U_j}{\partial x_i^2} - \mu \frac{\partial}{\partial x_j}\frac{\partial U_i}{\partial x_i} = -\mu \frac{\partial^2 U_j}{\partial x_i^2}$$

$$\boxed{\rho \frac{\partial U_j}{\partial t} + \rho U_i \frac{\partial U_j}{\partial x_i} = -\frac{\partial P}{\partial x_j} - \mu \frac{\partial^2 U_j}{\partial x_i^2} + \rho g_j}$$

# Navier-Stokes Equation IV

- Energy Conservation→Energy Equation

$$\underbrace{\rho c_{\mu}\frac{\partial T}{\partial t}}_{I}+\underbrace{\rho c_{\mu}U_{i}\frac{\partial T}{\partial x_{i}}}_{II}=\underbrace{-P\frac{\partial U_{i}}{\partial x_{i}}}_{III}+\underbrace{\lambda\frac{\partial^{2}T}{\partial x_{i}^{2}}}_{IV}\underbrace{-\tau_{ij}\frac{\partial U_{j}}{\partial x_{i}}}_{V}$$

I : Local energy change with time
II: Convective term
III: Pressure work
IV: Heat flux(diffusion)
V: Irreversible transfer of mechanical energy into heat

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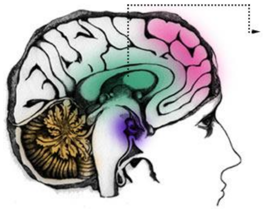
# Discretization



Analytical Equations → Discretization → Discretized Equations

- Discretization Methods
  - ✓ Finite Difference

    Straightforward to apply, simple, sturctured grids
  - ✓ Finite Element

    Any geometries
  - ✓ Finite Volume

    Conservation, any geometries

# Discretization

| Name of the Method | Process | Advantage | Disadvantage |
|---|---|---|---|
| **Finite-Difference Method (FDM)** | The method includes the assumption that the variation of the unknown to be computed is somewhat like a polynomial in x, y, or z so that higher derivatives are unimportant. | Straightforwardness and relative simplicity by which a newcomer in the field is able to obtain solutions of simple problems | Not suitable to solve problems with increasing degree of physical complexity such as flows at higher Reynolds numbers, flows around arbitrarily shaped bodies, and strongly time-dependent flows |

# Discretization

| Name of the Method | Process | Advantage | Disadvantage |
|---|---|---|---|
| **Finite element Method (FEM)** | It finds solutions at discrete spatial regions (called elements) by assuming that the governing differential equations apply to the continuum within each element. | ▪ Successful in solid mechanics applications.<br><br>▪ Their introduction and ready acceptance in fluid mechanics were due to relative ease by which flow problems with complicated boundary shapes could be modeled, especially when compared with FDMs. | ▪ More complicated matrix operations are required to solve the resulting system of equations<br><br>▪ Meaningful variational formulations are difficult to obtain for high Reynolds number flows<br><br>▪ Variational principle-based FEM is limited to solutions of creeping flow and heat conduction problems |

# Discretization

| Name of the Method | Process | Advantage | Disadvantage |
| --- | --- | --- | --- |
| **Spectral Method** | The approximation is based on expansions of independent variables into finite series of smooth functions. | It can be easily combined with standard FDMs. | ▪ Their relative complexity in comparison with standard FDMs<br>▪ Implementation of complex boundary conditions appears to be a frequent source of considerable difficulty |

# Discretization

| Name of the Method | Process | Advantage | Disadvantage |
|---|---|---|---|
| **Finite Volume Method (FVM)** | ▪ Domain is divided into a number of non-overlapping control volumes<br>▪ The differential equation is integrated over each control volume<br>▪ Piecewise profiles expressing the variation of the unknown between the grid points are used to evaluate the required integrals | Physical soundness | Not as straightforward as FDM |

# FVM-I

## General Form of Navier-Stokes Equation

$$\frac{\partial(\rho\Phi)}{\partial t}+\frac{\partial}{\partial x_i}\left(\rho U_i\Phi-\Gamma_\Phi\frac{\partial\Phi}{\partial x_i}\right)=q_\Phi \qquad \Phi=\left\{1,U_j,T\right\}$$

**Local change with time** **Flux** **Source**

**Integrate over the Control Volume(CV)**

$$\int_V \frac{\partial}{\partial x_i}\Phi dV=\int_S \Phi\cdot n_i dS$$

## Integral Form of Navier-Stokes Equation

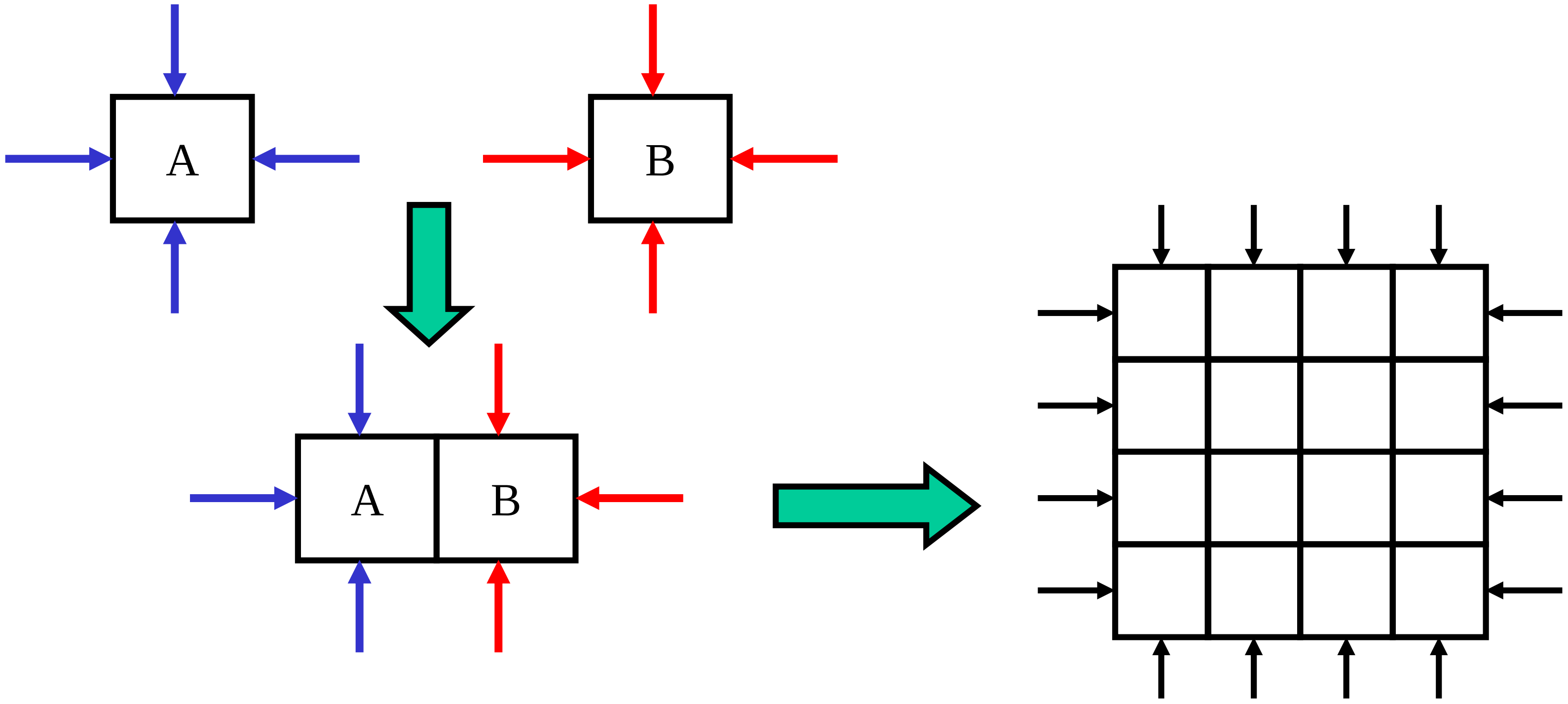
$$\int_V \frac{\partial(\rho\Phi)}{\partial t}dV+\int_S\left(\rho U_i\Phi-\Gamma\frac{\partial\Phi}{\partial x_i}\right)\cdot n_i dS=\int_V q_\Phi dV$$

**Local change with time in CV** **Flux Over the CV Surface** **Source in CV**

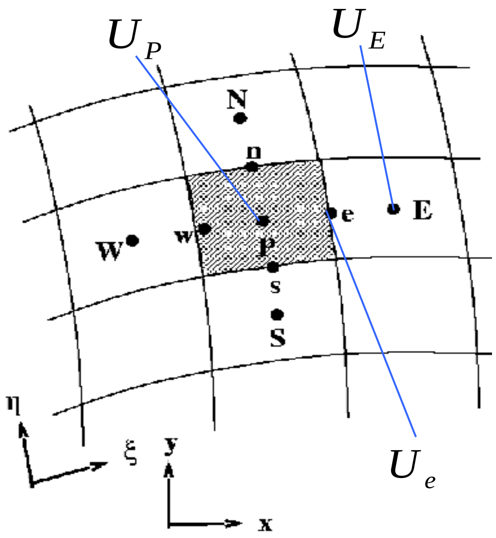
# FVM-II

## Conservation of Finite Volume Method

$$\int_V \frac{\partial(\rho\Phi)}{\partial t}dV + \int_S \left(\rho U_i \Phi - \Gamma \frac{\partial \Phi}{\partial x_i}\right)\cdot n_i dS = \int_V q_\Phi dV$$



# FVM-III




**Approximation of Volume Integrals**

$$m = \int_{V_i} \rho \, dV \approx \rho_p V; \qquad mu = \int_{V_i} \rho_i u_i dV \approx \rho_P u_P V$$

**Approximation of Surface Integrals ( Midpoint Rule)**

$$\int_{V_i} \nabla P \, dV = \oint_{S_i} P \, dS \approx \sum_k P_k \, S_k \qquad k = n, s, e, w$$

**Interpolation**

**Upwind** $$U_e = \begin{cases} U_P & \text{if } (\vec{U} \cdot \vec{n})_e > 0 \\ U_E & \text{if } (\vec{U} \cdot \vec{n})_e < 0 \end{cases}$$

**Central** $$U_e = U_E \lambda_e + U_P (1 - \lambda_e) \qquad \lambda_e = \frac{x_e - x_P}{x_E - x_P}$$

# Discretization of Cont. Eqn

## One Control Volume

$$a_P u_P + a_N u_N + a_S u_S + a_W u_W + a_E u_E = 0$$

## Whole Domain

$$\begin{bmatrix}
a_{11} & a_{12} & & & a_{1l} & & & & & \\
a_{21} & a_{22} & a_{23} & & & a_{2,l+1} & & & & \\
 & . & . & . & & & . & & & \\
 & & . & . & . & & & . & & \\
a_{k1} & & & . & . & . & & & a_{l-1,n-1} & \\
 & a_{k+1,2} & & & . & . & . & & & a_{l,n} \\
 & & . & & & . & . & . & & \\
 & & & . & & & . & . & . & \\
 & & & & a_{n-1,n-k-1} & & & a_{n-1,n-2} & a_{n-1,n-1} & a_{n-1,n} \\
 & & & & & a_{n,n-k} & & & a_{n,n-1} & a_{nn}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ . \\ . \\ . \\ u_{n-1} \\ u_n \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ . \\ . \\ . \\ 0 \\ 0 \end{bmatrix}$$

# Discretization of NS Eqn

- FV Discretization of Incompressible N-S Equation

$$Mu_h = 0$$

$$\Omega \frac{du_h}{dt} + C(u_h)u_h + Du_h - Mq_h = 0$$

Unsteady Convection Diffusion Source

- Time Discretization

$$\frac{du_h^{n+1}}{dt} = \begin{cases} f(u_h^n) & \text{Explicit} \\ f(u_h^n, u_h^{n+1}) & \text{Implicit} \end{cases}$$

# Introduction

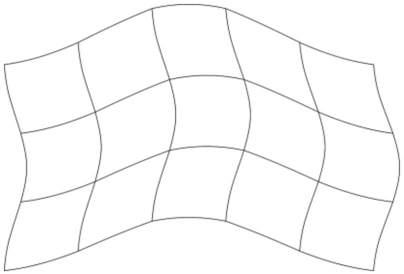
- What is Computational Fluid Dynamics(CFD)?
- Why and where use CFD?
- Physics of Fluid
- Navier-Stokes Equation
- Numerical Discretization
- Grids
- Boundary Conditions
- Numerical Staff

# Grids

- Structured Grid

  + all nodes have the same number of elements around it

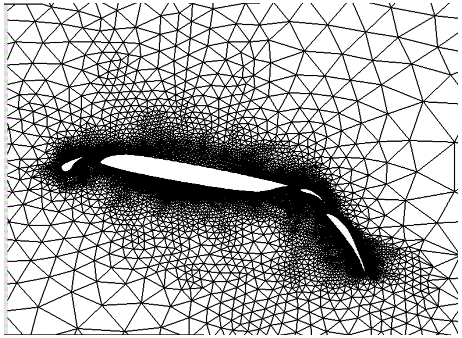
  – only for simple domains



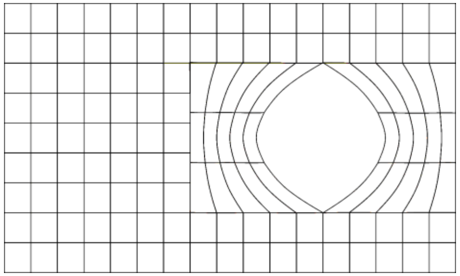
- Unstructured Grid

  + for all geometries

  – irregular data structure



- Block Structured Grid



# Introduction

- What is Computational Fluid Dynamics(CFD)?
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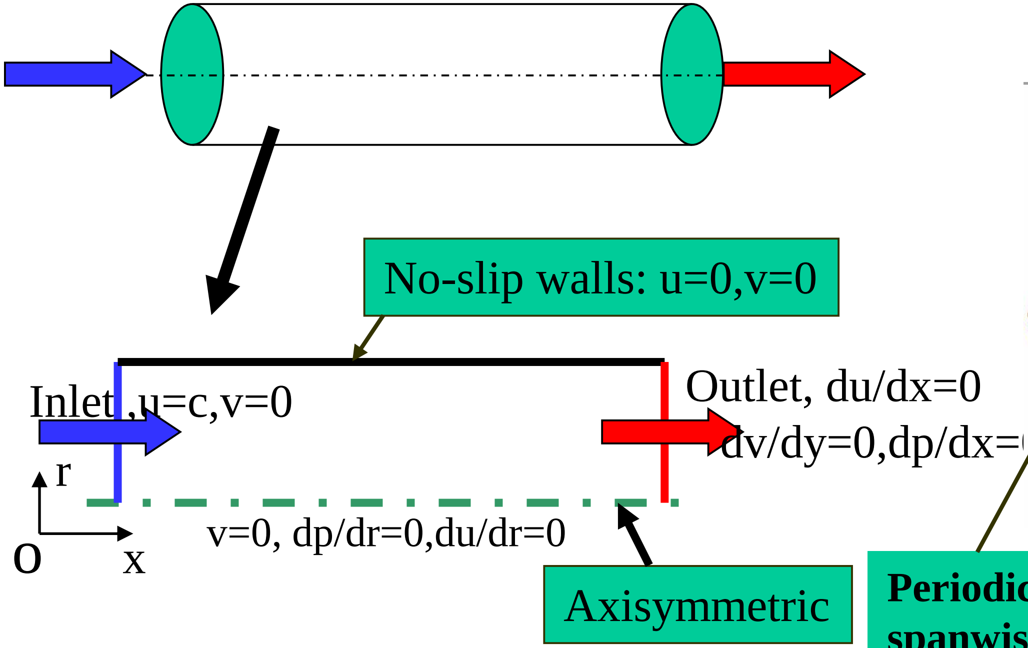
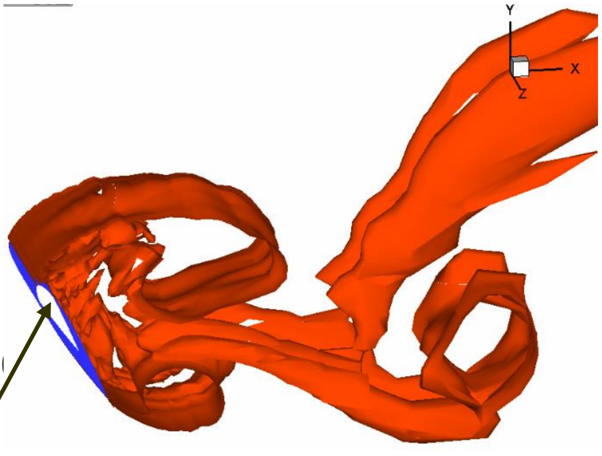
# Boundary Conditions

➢ Typical Boundary Conditions

No-slip(Wall), Axisymmetric, Inlet, Outlet, Periodic



Periodic boundary condition in spanwise direction of an airfoil

# Contents

# Solvers and Numerical Staff

- Solvers
  - ✓ Direct: Cramer's rule, Gauss elimination, LU decomposition
  - ✓ Iterative: Jacobi method, Gauss-Seidel method, SOR method

- Numerical Parameters
  - ✓ Under relaxation factor, convergence limit, etc.
  - ✓ Multigrid, Parallelization
  - ✓ Monitor residuals (change of results between iterations)
  - ✓ Number of iterations for steady flow or number of time steps for unsteady flow
  - ✓ Single/double precisions

| Criteria | Detail | Examples |
|---|---|---|
| **order** | The order of a PDE is determined by the highest-order partial derivative present in that equation | First order: $\partial\phi/\partial x - G\,\partial\phi/\partial y = O$<br>Second order: $\partial^2\phi/\partial x^2 - \phi\,\partial\phi/\partial y = O$<br>Third order: $[\partial^3\phi/\partial x^3]^2 + \partial^2\phi/\partial x\partial y + \partial\phi/\partial y = O$ |
| **linearity** | If the coefficients are constants or functions of the independent variables only, then Eq. is *linear. If* the coefficients are functions of the dependent variables and/or any of its derivatives of either lower or same order, then the equation is *nonlinear.* | $a\,\partial^2\phi/\partial x^2 + \mathrm{b}\,\partial^2\phi/\partial x\partial y + c\,\partial^2\phi/\partial y^2 + d = O$ |

# Classification of PDEs

Linear second-order PDEs: **elliptic, parabolic, and hyperbolic**.
The general form of this class of equations is:

$$a\frac{\partial^2 \phi}{\partial x^2} + b\frac{\partial^2 \phi}{\partial x \partial y} + c\frac{\partial^2 \phi}{\partial y^2} + d\frac{\partial \phi}{\partial x} + e\frac{\partial \phi}{\partial y} + f\phi + g = 0$$

where coefficients are either constants or functions of the independent variables only.
The three canonical forms are determined by the following criteria:

- $b^2 - 4ac < 0$ *elliptic*
- $b^2 - 4ac = 0$ *parabolic*
- $b^2 - 4ac > 0$ *hyperbolic*

# Classification of PDEs

| PDE | Example | Explanation |
|---|---|---|
| **Elliptic** | Laplace's equation: $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$ Poisson's equation: $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = g(x, y)$$ | In elliptic problems, the function f(x, y) must satisfy both, the differential equation over a closed domain and the boundary conditions on the closed boundary of the domain. |
| **Parabolic** | Heat conduction $$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}$$ | In parabolic problems, the solution advances outward indefinitely from known initial values, always satisfying the known boundary conditions as the solution progresses. |
| **Hyperbolic** | Wave equation $$\frac{\partial^2 \phi}{\partial t^2} = \gamma^2 \frac{\partial^2 \phi}{\partial x^2}$$ | The solution domain of hyperbolic PDE has the same open-ended nature as in parabolic PDE. However, two initial conditions are required to start the solution of hyperbolic equations in contrast with parabolic equations, where only one initial condition is required. |

# Classification of N-S eqn

The complete Navier–Stokes equations in three space coordinates (x, y, z) and time (t) are a system of three nonlinear second-order equations in four independent variables. So, the normal classification rules do not apply directly to them. Nevertheless, they do possess properties such as *hyperbolic, parabolic, and elliptic:*

| | |
|---|---|
| **Hyperbolic Flows** | **• Unsteady, inviscid compressible flow. A compressible flow can sustain sound and shock waves, and the Navier–Stokes equations are essentially hyperbolic in nature.**<br>**• For steady inviscid compressible flows, the equations are hyperbolic if the speed is supersonic, and elliptic for subsonic speed.** |

# Classification of N-S eqn

| Parabolic Flows | Elliptic Flows | Mixed Flows |
|---|---|---|
| •The boundary layer flows have essentially parabolic character. The solution marches in the downstream direction, and the numerical methods used for solving parabolic equations are appropriate. | • The subsonic inviscid flow falls under this category.<br>•If a flow has a region of recirculation, information may travel upstream as well as downstream. Therefore, specification of boundary conditions only at the upstream end of the flow is not sufficient. The problem then becomes elliptic in nature. | There is a possibility that a flow may not be characterized purely by one type. For example, in a steady transonic flow, both supersonic and subsonic regions exist. The supersonic regions are hyperbolic, whereas subsonic regions are elliptic. |

# Initial and BC

The initial and boundary conditions must be specified to obtain unique numerical solutions to PDEs:

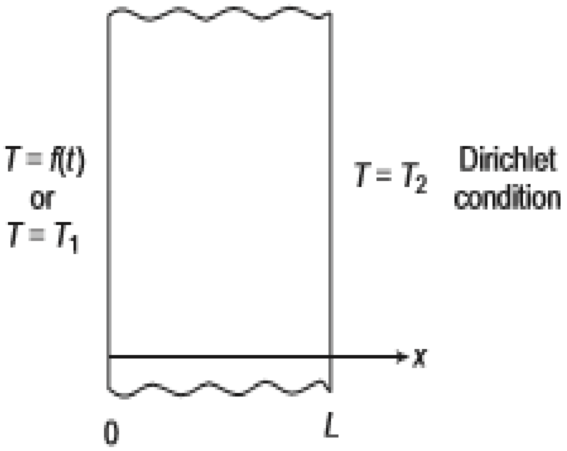
Following Eq. depicts a problem in which the temperature within a large solid slab having finite thickness changes in the x-direction as a function of time till steady state (corresponding to $t \rightarrow \infty$ ) is reached:

$$\frac{\partial T}{\partial t} = \gamma \frac{\partial^2 T}{\partial x^2}$$

**1. Dirichlet Conditions (First Kind):**

The values of the dependent variables are specified at the boundaries in the figure:

- Boundary Conditions of first kind can be expressed as
  B.C. 1 $T=f(t)$ *or* $T1$ *at* $x=0$
  $t > 0$
  *B.C.2* $T= T2$ *at* $x=L$
- Initial Condition
  $T= f(x)$ *at* $t= 0$ $0<= x =<L$
  *or* $T= T0$



# Initial and BC

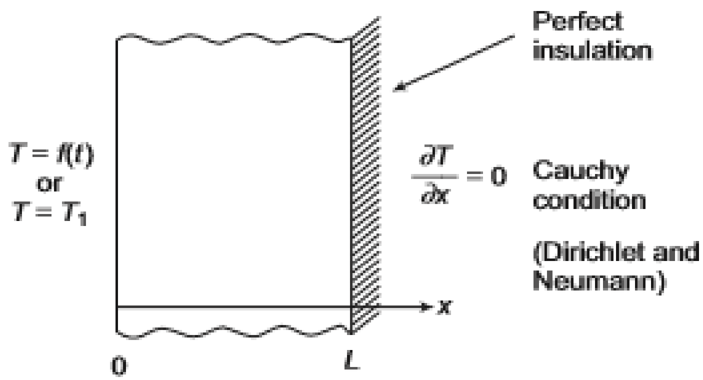
**2. Neumann Conditions (Second Kind)**

The derivative of the dependent variable is given as a constant or as a function of the independent variable on one boundary:

$$\frac{\partial T}{\partial x} = 0 \ldots at \ldots x = L \ldots and \ldots t \geq 0$$

This condition specifies that the temperature gradient at the right boundary is zero (insulation condition).

**Cauchy conditions:** A problem that combines both Dirichlet and Neumann conditions is considered to have Cauchy conditions:



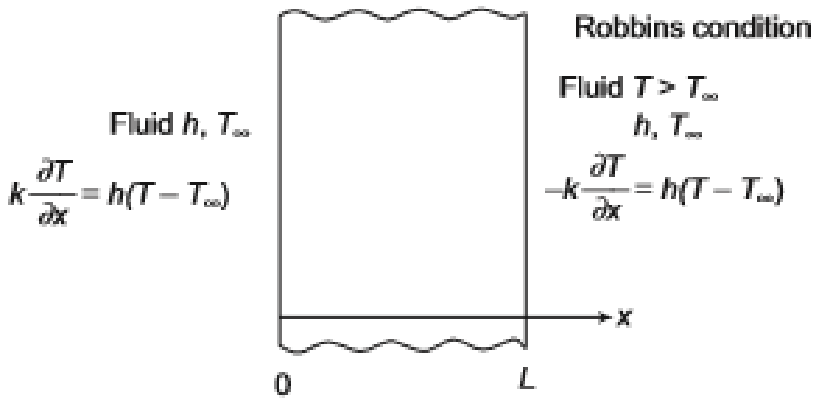

***hy conditions***

# Initial and BC

**3. Robbins Conditions (Third Kind)**

The derivative of the dependent variable is given as a function of the dependent variable on the boundary.

For the heat conduction problem, this may correspond to the case of cooling of
a large steel slab of finite thickness *"L"* by water or oil, the heat transfer coefficient *h* being finite:



# Initial and Boundary value probs

On the basis of their initial and boundary conditions, PDEs may be further classified into initial value or boundary value problems.

- **Initial Value Problems:**

In this case, at least one of the independent variables has an open region. In the unsteady state heat conduction problem, the time variable has the range $0 \leq t \leq \infty$, where no condition has been specified at $t = \infty$; therefore, this is an initial value problem.

- **Boundary Value Problems:**

When the region is closed for all independent variables and conditions are specified at all boundaries, then the problem is of the boundary value type. An example of this is the three-dimensional steady-state heat conduction (with no heat generation) problem, which is mathematically represented by the equation:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = 0$$


Thank You!