Recursive Functions

Function Calling Itself

- Certain problems naturally lend themselves to recursion.
- Eg. factorial. X^n, sum of n numbers.
- Solves a problem by
  1. Reducing it an instance of the same problem with smaller input.
  2. Having a smallest instance that can be computed directly without making calls to itself.
- So it has two parts:
- **Base case**
- **Recursive step**

Recursive Functions

Connection with Induction

- Induction has a **base case** and **inductive step**
- Basis is directly proved.
- Inductive step relies on the assumption that smaller instance is already solved.
- Eg. prove $1 + 2 + 2^2 + \ldots + 2^n = 2^{n+1} - 1$
- $2^n < n!$ for $n \geq 4$.

Recursive Functions

Connection with Induction

Weak induction

- Base case: Given $P(n_0)$ is true.
- Induction step: for all $k \geq n_0$, $P(k) \implies P(k+1)$

Strong induction

- Base case: Given $P(n_0)$ is true.
- Induction step: for all $k \geq n_0$ $(P(n_0) \wedge P(n_0+1) \wedge \ldots \wedge P(k)) \implies P(k+1)$

- Strong Induction can be derived from Weak Induction.
- Reaching the top from the middle of a ladder is impossible unless all rungs of the ladder can be reached from the bottom.

Recursive Functions

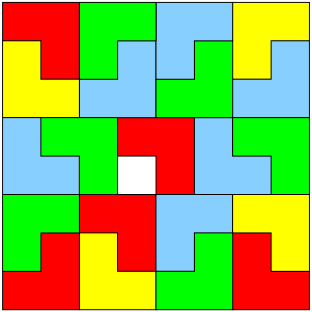
Strong Induction

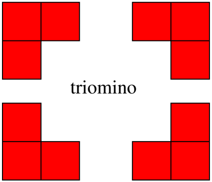
- Why strong induction is necessary?
- Consider of breaking a chocolate: $mn - 1$ breakings necessary for a mn sized chocolate.
- $P(1)$ is true: no breaking is necessary for a single piece.
- For $P(mn)$: breaking once, we get pieces of size $m_0 n_0$ and $m_1 n_1$.
- To prove the hypothesis: both $P(m_0 n_0)$ and $P(m_1 n_1)$ are needed.
- Prove by Weak Induction not possible.
- Recursion is based on the idea of SI.

Recursive Functions

Strong Induction

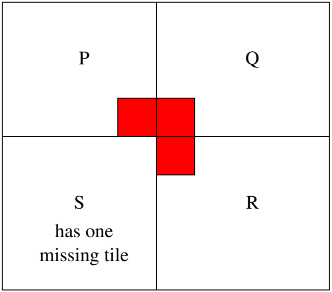
- For a +ve integer n, $2^n \times 2^n$ chess board with any 1 square removed can be tiled with right triominoes.





Recursive Functions

Strong Induction



- **Basis**: $P(1)$ true.
- **Hypothesis**: $P(k)$ be true.
- **Induction Step**: consider $P(k+1)$.
  - Divide C into 4 boards of size $2^k \times 2^k$.
  - Place a triomino so it covers one tile each of P, Q, R.
  - Now apply induction on each of the 4 boards.