Sketch of the solution of Tiling Problem

Note : This was a difficult problem compared to problems of other lab tests. The realization came too late.

Let $T(n)$ be the number of different ways to tile (or cover) a $2 \times n$ area using tiles of dimensions : 1×1, 2×1, 2×2. The additional rule that is to be followed during covering is that each tile of 1×1 dimension which appears in a covering must have at least one 1×1 tile neighboring/adjacent to it.

First attempt : Following the lines of reasoning used in the analysis of quiz 3 problem, One comes up with the following recursive formulation for $T(n)$.

$$T(n) = \begin{cases} 1 & \text{if } \; n = 0 \\ 2 & \text{if } \; n = 1 \\ 2 * T(n-1) + 4 * T(n-2) & \text{for } n > 1 \end{cases}$$

The intuition used here is that, due to the additional rule, each 1×1 tile can be paired up with a unique neighboring 1×1 tile and this pair is *effectively* equivalent to a 2×1 tile. This is a very good attempt. In fact it is very impressive to see that most of the students were able to come up with this formulation. This confirms that they have developed the skill of using recursion as a powerful tool to solve various computational problem. But, ..., it is not fully correct. The first mismatch occurs at $n = 5$. Since the level of the problem was considerably high, it is decided that students who are using the above recurrence or have made some **meaningful attempt** will be considered favorably.

I am providing below a tiling which is not captured by the above equation. Thanks to P. Arun Kumar and Vikas Marda for this example. Then I shall provide sketch of the solution. If you find solution difficult to follow, do not worry, such a time consuming question which involves lot of case analysis will not be asked in the end semester exam. But you should at least glance though it to get its overview.

The intuition on which our first attempt was based is not correct as shown by the following covering of a 2×5 rectangular area. You can see that for any

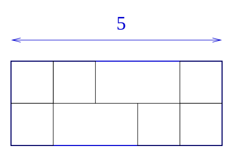

Figure 1: a covering of 2×5 area missed during the first attempt

pairing of 1×1 tiles, there will be at least one unpaired tile. In fact there will be two unpaired 1×1 in the above case. This is a valid tiling but is not counted in the abive recurrence. The main reason for ignoring this case is our short sightedness. We should not jump to conclusion very quickly. We should explore more...

The solution : After developing some insight from Figure 1, and some thinking, one can realize the following fact. Having covered the area from left to right upto a certain position, we need to know the information about the rightmost tile(s). In particular, we need to know whether there are one, two or zero tiles of dimension 1×1 at the right most place. Based on this idea, we realize that we need recursive formulation for three cases as shown in Figure 2. In particular,

- $D(n)$: The number of possible covering of an $2 \times n$ area when there is no 1×1 tile immediately left of it. Note that $T(n) = D(n)$.
- $C(n)$: The number of possible covering of an $2 \times n$ area when there is one 1×1 tile immediately left of it.
- $E(n)$: The number of possible covering of an $2 \times n$ area when there are two 1×1 tile immediately left of it.

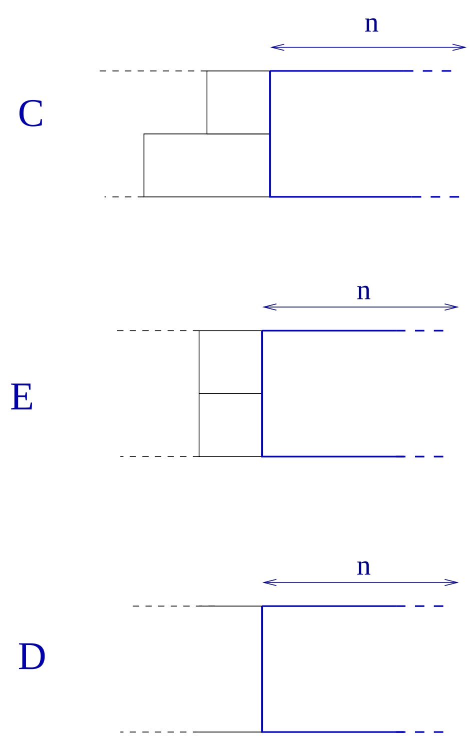

Figure 2: Classification of Tilings of rectangular area

Now when we start covering the area of any of the three types as mentioned above, we get the following non-rectangular areas as shown in Figure 3. More specifically, these areas are obtained by adding one 1×1 tile to corner of a rectangle of height 2. These shapes resemble a toy *wagon*, and henceforth we shall call them *wagon* shaped area. Please see the Figure 3 and the notations *Top* and *Front*. It can be seen that, for the degenerate case of $n = 0$, $M(n) = N(n) = 1$, whereas $O(n)$ is not defined for $n = 0$. In fact $O(n)$ is defined for $n > 0$ only.

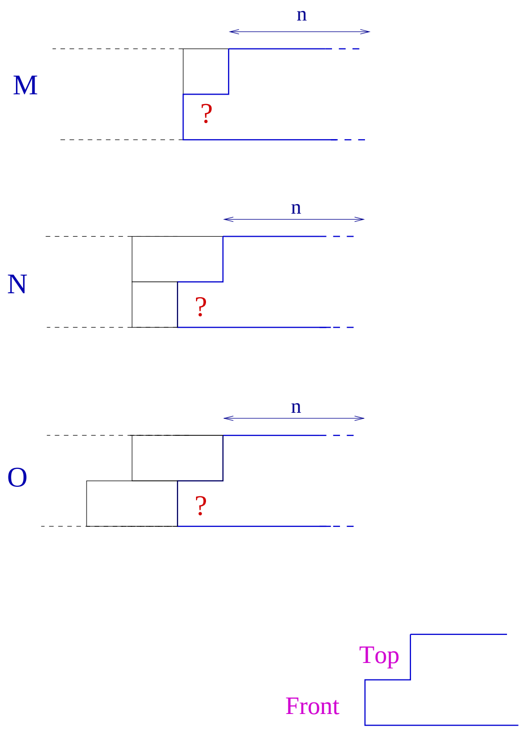

Figure 3: Classification of Tilings of wagon shaped area

We use one more terminology here.

- $M(n)$: the number of possible covering of a wagon shaped $2 \times n$ area when there is a 1×1 tile at the top.
- $N(n)$: the number of possible covering of a wagon shaped $2 \times n$ area when there is a 2×1 tile at the top and a 1×1 tile in the front.
- $O(n)$: the number of possible covering of a wagon shaped $2 \times n$ area when there is a 2×1 tile at the top and the tile in front position is not of dimension 1×1.

Now based on which tile will be placed at ? location, there are following recursive formulations for $M(n), N(n), O(n)$.

$$M(n) = \begin{cases} 1 & \text{if } \ n = 0 \\ E(n) + N(n-1) & \text{if } \ n > 0 \end{cases}$$

$$N(n) = \begin{cases} 1 & \text{if } n = 0 \\ C(n) + O(n-1) & \text{if } n > 0 \end{cases}$$

$$O(n) = \begin{cases} 1 & \text{if } n = 1 \\ M(n-1) + O(n-1) & \text{if } n > 1 \end{cases}$$

We provide explanation for the recursive formulation of $M(n)$ (formulations for $N(n), O(n)$ are along similar lines). At the location ?, there can be either a 1×1 tile or a 2×1 tile. Note that placing 1×1 tile at location ? does not violate the rule mentioned in the problem. Now if we place a 1×1 tile at location ?, then the corresonding covering is same as that of a $2 \times n$ rectangular area with two 1×1 tile on the left (this is precisely $E(n)$). If we place 2×1 tile at ?, then the corresonding covering is same as that of a $2 \times (n-1)$ *wagon* shape (but inverted) with 1×1 tile in front position (this is precisely $N(n-1)$). Hence the recursive formulation of $M(n)$ follows.

Now we give the formulation for $C(n), D(n)$, and $E(n)$ as follows.

$$C(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ D(n-1) + 2 * D(n-2) + C(n-2) + M(n-1) & \text{if } n > 1 \end{cases}$$

$$D(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ D(n-1) + 2 * D(n-2) + 2 * C(n-2) + E(n-1) & \text{if } n > 1 \end{cases}$$

$$E(n) = \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n = 1 \\ D(n-1) + 2 * D(n-2) + 2 * N(n-2) + E(n-1) & \text{if } n > 1 \end{cases}$$

The attached program for computing the number of possible coverings/tilings of an $2 \times n$ area is based on the above recursive formulation.

Output

- n=1; T(n) = D(n) = 2;
- n=2; T(n) = D(n) = 8;
- n=3; T(n) = D(n) = 24;
- n=4; T(n) = D(n) = 80;
- n=5; T(n) = D(n) = 258.
  Note that the recursive solution of our first attempt will miss two coverings which are captured in the above solution. One covering is the covering shown in Figure 1, and another one is mirror image of it.

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