Optics of GW detectors

Review of optics

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Outline

1. Electromagnetic fundamentals
2. Describing optical elements
3. Michelson interferometer
4. Fabry-Perot cavity
5. Higher-order transverse modes
Detecting optical signals

- Sinusoidal optical signals characterized by amplitude/power, frequency, phase, and polarization
- Photodetector (PD): Produces current proportional to incident optical power $I_{ph} = \mathcal{R}P_{opt}$, $\mathcal{R}=$PD responsivity
- PDs are insensitive to phase of optical waves
- How to measure phase then? Using an interferometer
What is an interferometer?

- Interferometers convert phase to intensity/power.
- In GW detector context:
  - Optical phase difference $\propto$ differential strain: $\delta \phi = G \delta L$.
  - Converts $\delta \phi$ to intensity/power.
  - Goal is to make $G$ large.

$$
\Phi_m \rightarrow I_{out1} = f_1(\phi_m, \phi_r) \\
\Phi_r \rightarrow I_{out2} = f_2(\phi_m, \phi_r)
$$

- $\phi_m =$ phase to be measured, $\phi_r =$ reference phase.
Michelson interferometer layout

- Consists of light source, two arms with end mirrors, and beamsplitter

- Michelson interferometer from 1881; simplified optical layout
Maxwell’s equations

- Classical light is electromagnetic phenomena; described by Maxwell’s equations
  - Faraday’s law: \( \nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{B}(\vec{r}, t) \)
  - Ampere-Maxwell law: \( \nabla \times \vec{H}(\vec{r}, t) = \frac{\partial}{\partial t} \vec{D}(\vec{r}, t) + \vec{J}(\vec{r}, t) \)
  - Gauss’s laws: \( \nabla \cdot \vec{D}(\vec{r}, t) = \rho(\vec{r}, t) \) and \( \nabla \cdot \vec{B}(\vec{r}, t) = 0 \)
  - Constitutive relations \( \vec{D} = \epsilon \vec{E} \) and \( \vec{B} = \mu \vec{H} \) encode medium properties
- Harmonic solutions: \( \vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\omega t + \phi(\vec{r})) = \text{Re}[\vec{E}_0 e^{j\phi(\vec{r})} e^{j\omega t}] \)
  \( \vec{E}_0 e^{j\phi(\vec{r})} \) is called a phasor
Phasor representation

- Complex number, represented as a vector in complex plane
- Time-domain:
  \[ E \cos(\omega t + \phi) \rightarrow E e^{j\phi} = \underline{E} \]: Phasor
- Phasor: \[ E \rightarrow \text{Re}[E e^{j\omega t}] \]: Time-domain
- Exercise: Obtain phasor form of
  \[ \hat{x} \cos(\omega t - kz) + \hat{y}2 \sin(\omega t - kz) \]

E. D. Black and R. N. Gutenkust, AJP, 71(4), 2003
Describing optical waves: Plane wave description

- From Maxwell’s equations we obtain wave equation
  \[ \nabla^2 \vec{E} + \omega^2 \mu \varepsilon \vec{E} = 0 \]

- Optical waves propagating in \( z \)-direction; \( \vec{E}(\vec{r}) = \vec{E}_T(x, y)A e^{-jkz} \)
  - \( k = \frac{\omega}{c} = \frac{2\pi}{\lambda} \) is phase constant
  - \( \vec{E}_T(x, y) \) = transverse field distribution
  - **Plane wave**: \( \vec{E}_T(x, y) \) independent of \( x \) and \( y \) coordinates
  - Longitudinal part \( Ae^{-jkz} \) is a complex number at each \( z \)
  - \( \vec{E}_T \) determines **polarization** of wave

- Normalize such that \( |A|^2 \) is optical power
Polarization of light

- Defined as orientation of electric field vector $\vec{E}$ in space
  - Linear polarization: $\vec{E}$ orientation constant with time
  - Elliptical polarization: $\vec{E}$ orientation varies with time
  - Jones vector: $\begin{pmatrix} A_x \\ A_y \end{pmatrix}$

- Optical elements such as quarter-wave and half-wave plates can be used to change polarization

G. R. Fowles, Introduction to Modern Optics
Describing mirrors

- Mirrors are used extensively in GW detectors and other optical systems.
- Incident light partially reflected and transmitted by mirror.
- Flexibility to choose phase of reflection and transmission coefficients; \( \phi_r = \pi/2 \) or \( \pi \)

Mirror matrix is unitary

\[ M = \begin{pmatrix} jr & t \\ t & jr \end{pmatrix} \quad MM^\dagger = I \]

\[ \begin{align*}
E_1^+ &\rightarrow E_2^+ \\
E_1^- &\leftarrow E_2^-
\end{align*} \]

\[ \begin{align*}
E_1^- &= rE_2^+ + t'E_2^- \\
E_2^+ &= tE_1^+ + r'E_2^-
\end{align*} \]

- \(|r| = |r'| \) and \(|t| = |t'| \)
- \( r^*t' + t^*r' = 0 \) and \( |r|^2 + |t|^2 = 1 \)
- \( R = |r|^2 \) reflectivity and \( T = |t|^2 \) transmittivity
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\[ E_1^+ \rightarrow E_2^+ \]
\[ E_1^- \leftarrow E_2^- \]
\[ r, t \quad r', t' \]

- $E_1^- = rE_1^+ + t'E_2^-$
- $E_2^+ = tE_1^+ + r'E_2^-$

$|r| = |r'|$ and $|t| = |t'|$

$|r|^2 + |t|^2 = 1$

$R = |r|^2$ reflectivity and $T = |t|^2$ transmittivity
Describing optical elements

Reflection and transmission coefficients

- Depend upon polarization of incident light, angle of incidence w.r.t. normal to interface, and refractive index on two sides of interface
- TE case: Electric field vectors are perpendicular to plane of incidence
- Coefficients can be derived by applying boundary conditions

![Diagram showing optical elements and reflection coefficients](image)

G. R. Fowles, Introduction to Modern Optics
Boundary conditions

At interface, vectors field vectors satisfy following conditions

- Tangential $E$-field and normal $B$-field are continuous across boundary
- Tangential $H$-field and normal $D$-field are discontinuous by amount of current and charge densities respectively

In reflection coefficient calculation for TE case

- $E + E' = E''$
- $-H \cos(\theta) + H' \cos(\theta) = -H'' \cos(\phi)$
- $E/H = \eta/\sqrt{\varepsilon}$

G. R. Fowles, Introduction to Modern Optics
TM case: Magnetic field vectors are perpendicular to plane of incidence

Coefficients can be derived by applying boundary conditions

\[
r_p = \frac{-n \cos \theta + \cos \phi}{n \cos \theta + \cos \phi}
\]

G. R. Fowles, *Introduction to Modern Optics*
Zero reflection in TM case when light is incident at **Brewster's angle**

- This plot: $n_1 = 1$ (air) and $n_2 = 1.5$. What happens if $n_1 > n_2$?
Describing optical elements

Describing lossy mirrors

- Real mirrors are lossy due to absorption by mirror material
- Reflectance + Transmittance + loss = 1, $|r|^2 + |t|^2 + L = 1$
- Further complication due to fluctuation-dissipation theorem which states that loss is accompanied by additional noise injected into system
- $\epsilon =$ absorption coefficient

Describing beamsplitter

- I/O relation described by same matrix $M$
- Types: polarizing and non-polarizing
- Common 50:50 beamsplitter:
  \[ B = \frac{1}{\sqrt{2}} \begin{pmatrix} j & 1 \\ 1 & j \end{pmatrix} \]
- Delay: $A(L) = A(0)e^{-j k_0 n L}$, accumulates phase delay $k_0 n L$ w.r.t $z = 0$
Layout and analysis of Michelson interferometer

- Beamsplitter splits laser light into two parts; one travels towards $M_X$ other towards $M_Y$
- After reflection at mirrors $M_{x,y}$, beams recombine at beamsplitter
- $A_2 = \frac{j}{\sqrt{2}} A_1$, $A_5 = jr_Y e^{-j2kL_Y} A_2$
- $A_6 = \frac{1}{\sqrt{2}} A_1$, $A_9 = jr_X e^{-j2kL_X} A_6$

**Interferometer output amplitudes**

**ASYM port:** $A_{ASYM} = -\frac{1}{2} \left( r_X e^{-j2kL_X} - r_Y e^{-j2kL_Y} \right) A_1$

**SYM port:** $A_{SYM} = \frac{j}{2} \left( r_X e^{-j2kL_X} + r_Y e^{-j2kL_Y} \right) A_1$
Matrix analysis of Michelson interferometer

- Input vector at port 1: $\vec{\psi} = [A_1 \ 0]^T$
- Propagation+reflection+propagation towards beamsplitter described by matrix $P = \begin{pmatrix} jr_x e^{-j2kL_x} & 0 \\ 0 & jr_y e^{-j2kL_y} \end{pmatrix}$
- (Try) Multiply three matrices with input vector: $B^{-1} P B \vec{\psi}$ to get $A_{ASYM}$ and $A_{SYM}$
Effect of gravitational wave

- GW perturbs mirrors and induces changes in reflected light
- $A_5 \rightarrow r_Y e^{-j2kL_Y} e^{-j2kL_Y h(t)/2} A_2$, $h(t)$ induces phase modulation
- Harmonic GW, $h(t) = h_0 \cos(\omega_{gw} t)$ creates sidebands

$$A_5 = A_2(0) \left( 1 - \frac{jm}{2} e^{j\omega_{gw} t} - \frac{jm}{2} e^{-j\omega_{gw} t} \right)$$

- More about phase modulation later
**Understanding interferometer response**

- ASYM port amplitude: $A_{ASYM} = -\frac{1}{2} \left( r_X e^{-j2kL_X} - r_Y e^{-j2kL_Y} \right) A_1$
- Assume perfectly reflecting mirrors without loss: $r_X = r_Y = 1$
- ASYM port power $P_{ASYM} = P_{in} \sin^2(k\Delta L)$, $\Delta L = L_X - L_Y$
Operating in linear region

- Under GW perturbation, $L_X \rightarrow L_X + \delta l_X$ and $L_Y \rightarrow L_Y + \delta l_Y$
- Amplitude strain $h = \frac{\delta l_X - \delta l_Y}{L}$, $L = \frac{L_X + L_Y}{2}$ is avg. length
- $P_{ASYM} = P_{in} \sin^2(k\Delta L + khL)$. What should be $k\Delta L$ for operation in linear region?
- Expand $P_{ASYM}$ using Taylor series with $khL$ as perturbation

$$P_{ASYM} = P_{in} \sin^2(khL) + P_{in}khL \frac{\partial}{\partial(k\Delta L)} \sin^2(k\Delta L) + \cdots$$

- What value of $k\Delta L$ makes derivative maximum? (Ans. $\pi/4$)
- $P_{ASYM} \approx \frac{P_{in}}{2} (1 + 2khL)$; Laser intensity fluctuations swamps small signal $(khL)$ term $\implies$ Linear region: bad!
Null region operation

- At null point, $k \Delta L = 0$ so that $P_{ASYM} = P_{in} \sin^2(khL) \approx k^2 h^2 L^2$
- Since $h \ll 1$, $h^2 \ll 1$ makes detection a challenge
- Phasor analysis shows field exiting ASYM port is in quadrature ($\pi/2$) with respect to incident light
- Here beamsplitter and mirrors are assumed to provide $180^\circ$ phase shift upon reflection

E. D. Black and R. N. Gutenkust, AJP, 71(4), 2003
Signal extraction using lock-in

- Modulate carrier to generate sidebands at $\lambda_{mod}$
- Make FP cavity dark only to carrier fields (Schnupp asymmetry)
Signal extraction using lock-in

\[ k_\pm = \frac{\omega \pm \Omega}{c} = 2\pi \left( \frac{1}{\lambda} \pm \frac{1}{\lambda_{\text{mod}}} \right) \]

\[ t_\pm = i \sin \left[ 2\pi \left( \frac{\ell_x - \ell_y}{\lambda} \pm \frac{\ell_x - \ell_y}{\lambda_{\text{mod}}} \right) \right] e^{ik_\pm(\ell_x + \ell_y)} \]

\[ t_\pm = \mp i \sin \left[ 2\pi \left( \frac{\Delta \ell}{\lambda_{\text{mod}}} \right) \right] e^{i[(\omega \pm \Omega)/c](\ell_x + \ell_y)} \]

\[ P_{\text{out}} = P_{\text{in}} J_0^2(\beta) 4\pi^2 \frac{\ell^2}{\lambda} h^2 + 2P_{\text{in}} J_1^2(\beta) \sin^2 \left( 2\pi \frac{\Delta \ell}{\lambda_{\text{mod}}} \right) \]

\[ + 2P_{\text{in}} J_1^2(\beta) \sin^2 \left( 2\pi \frac{\Delta \ell}{\lambda_{\text{mod}}} \right) \cos \left( 2\Omega t + 8\pi \frac{\ell}{\lambda_{\text{mod}}} \right) \]

\[ + P_{\text{in}} J_0(\beta) J_1(\beta) 4\pi \frac{\ell}{\lambda} h \sin \left( 2\pi \frac{\Delta \ell}{\lambda_{\text{mod}}} \right) \]

\[ \times \cos \left( 2\Omega t + 4\pi \frac{\ell}{\lambda_{\text{mod}}} \right). \]
Mirror reflection mismatch

- In practice \( r_X, Y = r \pm \frac{\delta r}{2} \)
- \( P_{ASYM} = \frac{1}{4} \left[ \left( r^2 - \frac{\delta r^2}{4r^2} \right) \cos^2 (k\delta L) + \frac{\delta r^2}{4r^2} \right] P_{in} \)

G. Vajente, Chap. 3, Advanced interferometers and the search for gravitational waves
Fabry-Perot cavity: layout

- Formed by two mirrors, \( M_1 = \{ r_i, t_i \} \) and \( M_2 = \{ r_e, t_e \} \), \( t_e \approx 1 \)
- \(-r_i r_e e^{-j2kL} E_{FP}\) fed back to cavity

\[
E_{FP} = t_i E_{in} - r_i r_e e^{-j2kL} E_{FP} = \frac{t_i E_{in}}{1 + r_i r_e e^{-j2kL}}
\]

- Output field: \( E_{out} = t_e e^{-jKL} E_{FP} = \frac{t_i t_e E_{in}}{1 + r_i r_e e^{-j2kL}} \)
FP cavity: characterization

- At resonance, $e^{-j2kL} = -1$ and $P_{FP} = P_{in}\frac{(t_i t_e)^2}{(1-r_i r_e)^2} = G_{FP} P_{in}$
- Resonance condition implies multiple peaks spaced half-wavelength apart defining free-spectral range $FSR = \frac{c}{2L}$
- With detuning $\delta L$, $P_{FP} = \frac{t_i^2}{(1-r_i r_e)^2 + 4r_i r_e \sin^2(k\delta L)} P_{in}$
- $\delta L_{FWHM} = \frac{\lambda}{2\mathcal{F}}$, where $\mathcal{F} = \frac{\pi \sqrt{r_i r_e}}{1-r_i r_e}$ is cavity finesse
- $P_{FP} = \frac{G_{FP}}{1 + \left(\frac{2\mathcal{F}}{\pi}\right)^2 \sin^2(k\delta L)} P_{in}$
- Typical finesse values are $>50$
In this plot, $\mathcal{F} = 30$ and $r_i = 0.9$. Calculate $r_e$ and $t_i$.
FP cavity: reflection

\[ E_{\text{ref}} = j \frac{r_i + r_e (t_i^2 + r_i^2) e^{-j2kL}}{1 + r_i r_e e^{-j2kL}} E_{\text{in}} \]

In this plot, \( \mathcal{F} = 30 \) and \( r_i = 0.9 \). Calculate \( r_e \) and \( t_i \)

G. Vajente, Chap. 3, Advanced interferometers and the search for gravitational waves
FP cavity: reflection

- When $r_i = r_e$, at resonance, light is completely transmitted (critical coupling)
- When $r_i < r_e$, at resonance, light is reflected mostly but more importantly phase is highly sensitive to length variations (over coupling)
- To implement **power recycling**, FP cavities are operated in over coupling mode
Mirror motion

- Mirror motion due to GW perturbation results in sidebands
- Displacement $x(t) = x_0 \cos(\Omega_s t)$ yields phase shift $\phi_s = 2kx(t)$
- $E_4 = j r_e e^{-j\phi_s} E_3 \approx j r_e E_3 + r_e \phi_s E_3 = j r_e E_3 + k r_e x_0 \left(e^{j\Omega_s t} + e^{-j\Omega_s t}\right) E_3$
- Sideband amplitude at resonance: $E_4(f_s) = \frac{k x_0 E_3(0)}{1 - r_i r_e e^{-j2(\Omega_s/c)L}}$

$$E_{\text{ref}} = \left[ j \sqrt{G_{FP}} k r_e x_0 \frac{j t_i e^{-j(\Omega_s/c)L}}{1 - r_i r_e e^{-j2(\Omega_s/c)L}} \right] E_{\text{in}}$$
FP cavity: Frequency response

- For GW frequencies $\Omega_s L/c \ll 1$ so that $E_{ref} = -k r e x_0 G_{FP} \frac{1}{1 + j \frac{f_s}{f_p}} E_{in}$,

  where $f_p = \frac{c}{4 L F}$ is critical frequency

- This is **low-pass filter** transfer function with low-frequency gain of $k r e x_0 G_{FP}$ and 3-dB bandwidth $f_p$

- Since $f_p \propto F^{-1}$, high finesse leads to lower bandwidth (why?)

![Graph showing frequency response](image-url)
Simulating FP cavities using Finesse

- Finesse is a frequency-domain simulation tool for interferometric detectors
- Easy to use and free!
- Latest version 2.0 released
Higher-order transverse modes

Paraxial wave equation

- Practical optical beams are not plane waves; they are described by \textit{paraxial wave equation}
- \((\nabla^2 + k^2)E(x, y, z) = 0\) with \(E(x, y, z) = e^{ikz} A(x, y, z)\)
- Paraxial approximation: \(|\frac{\partial^2 A}{\partial z^2}| \ll \frac{2\pi}{\lambda} A\) gives equation \((\partial_x^2 + \partial_y^2 + 2jk\partial_z) A = 0\) describing propagation of beams

\[
A(r, z) = \frac{1}{\sqrt{1 + \frac{z^2}{z_R^2}}} e^{-\frac{x^2 + y^2}{2w(z)^2}} e^{-ik\frac{x^2 + y^2}{2R(z)}} e^{i \arctan \frac{z}{z_R}} e^{-ikz}
\]

\[
w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}} \\
R(z) = z \left(1 + \frac{z^2}{z_R^2}\right)
\]

\[
z_R = \frac{k w_0^2}{2} \\
\phi_G = - \arctan \frac{z}{z_R}
\]
Gaussian beams

- Circularly symmetric with minimum transverse width $w_0$ at $z = 0$ known as beam waist
- $w(z)$ grows with $z$; at $z = z_R$, Rayleigh distance, $w(z) = \sqrt{2}w_0$

\[
A(r, z) = \frac{1}{\sqrt{1 + \frac{z^2}{z_R^2}}} e^{-\frac{x^2+y^2}{w^2(z)}} e^{-ik\frac{x^2+y^2}{2zR(z)}} e^{i \arctan \frac{z}{z_R}} e^{-ikz}
\]

\[
w(z) = w_0 \sqrt{1 + \frac{z^2}{z_R^2}} \quad R(z) = z \left(1 + \frac{z^2}{z_R^2}\right)
\]

\[
z_R = \frac{kw_0^2}{2} \quad \phi_G = -\arctan \frac{z}{z_R}
\]
Higher-order transverse modes

- Different solutions (modes) of paraxial equation; not necessarily cylindrical symmetric
- Common modes: Hermite-Gaussian or transverse electro-magnetic modes (TEM$_{mn}$)

\[
\begin{align*}
\text{TEM}_{mn}(x, y, z) &= N_{mn}(z)e^{i k z}H_m\left(\frac{\sqrt{2}x}{w(z)}\right)H_n\left(\frac{\sqrt{2}y}{w(z)}\right) \\
&\quad e^{-i(n+m+1)\arctan(z/z_R)}e^{ik\frac{x^2+y^2}{2R(z)}}e^{\frac{-x^2+y^2}{w^2(z)}}
\end{align*}
\]

\[
N_{mn}(z) = \sqrt{\frac{2}{\pi w(z)^2}} \frac{2}{2^n m! n!}
\]

\[
H_n(t) = e^{t^2} \left(-\frac{d}{dt}\right)^n e^{-t^2}
\]
Higher-order transverse modes

TEM_{0,0}  

TEM_{1,0}  

TEM_{0,1}  

TEM_{1,1}
Resonators and beams

- Resonators cannot have plane surfaces (Why?); Stability of resonators depend on surface shapes

![TEM modes](image-url)