Guidance Theory and Applications (Lecture 2)

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Some Fundamental Results based on Kinematics

- Kinematics of interception between unguided moving objects
- The concept of miss-distance
- Interception and avoidance
Figure 4.1: *Engagement between two point objects*
The Relative Velocities

- Note that both missile and target speeds are constants.

- Also, the objects do not maneuver and so their flight path angles are also constants.

- Only quantities to change are LOS separation $R$ and LOS angle $\theta$.

\[ V_R = \dot{R} = V_T \cos(\alpha_T - \theta) - V_M \cos(\alpha_M - \theta) \]
\[ V_\theta = R\dot{\theta} = V_T \sin(\alpha_T - \theta) - V_M \sin(\alpha_M - \theta) \]
Dynamics of relative velocities

Differentiate relative velocity components w.r.t. time

\[ \dot{V}_R = -V_T \sin(\alpha_T - \theta)(-\dot{\theta}) + V_M \sin(\alpha_M - \theta)(-\dot{\theta}) \]

\[ = \dot{\theta} \left\{ V_T \sin(\alpha_T - \theta) - V_M \sin(\alpha_M - \theta) \right\} \]

\[ = \dot{\theta} V_\theta \]

\[ \dot{V}_\theta = V_T \cos(\alpha_T - \theta)(-\dot{\theta}) - V_M \cos(\alpha_M - \theta)(-\dot{\theta}) \]

\[ = -\dot{\theta} \left\{ V_T \cos(\alpha_T - \theta) - V_M \cos(\alpha_M - \theta) \right\} \]

\[ = -\dot{\theta} V_R \]
LOS Rate ...

\[
\dot{\theta} = \frac{\dot{V}_R}{V_\theta} = -\frac{\dot{V}_\theta}{V_R}
\]

\[\Rightarrow V_R \dot{V}_R = -V_\theta \dot{V}_\theta\]

\[\Rightarrow V_R^2 \dot{V}_R + V_\theta \dot{V}_\theta = 0\]

Integrating ...

\[V_R^2 + V_\theta^2 = c^2\]
Which is the equation of a circle of radius $c$ and center at $(0,0)$ in the $(V_\theta, V_R)$-space.

\[ V_R(t = 0) = V_{R0} \]
\[ V_\theta(t = 0) = V_{\theta 0} \]

\[ c = \sqrt{V_{R0}^2 + V_{\theta 0}^2} \]
Direction of movement

\[ R\dot{V}_R = V_\theta^2 \]
\[ R\dot{V}_\theta = -V_\theta V_R \]

\[ \dot{V}_R > 0 \text{ always} \]
Collision condition

Collision triangle

☐ A1 and A2 are stationary points.
☐ At A1 and A2,

\[ V_\theta = 0 \]

☐ since \( R > 0 \) always,

\[ \dot{V}_R = 0 \]
\[ \dot{V}_\theta = 0 \]
At Point A1

Which implies that the LOS separation between the two objects shrinks at a constant rate if the state is on A1. Hence, after a finite time the two objects collide with each other. Thus, the collision condition is given by,

\[ V_{\theta} = 0 \quad V_R < 0 \]
LOS Angle

This implies that the LOS angle $\theta$ remains constant. So the LOS does not rotate in space.

Hence, the collision condition is satisfied when the LOS separation shrinks with time but the LOS angle does not vary.

This is the condition for the COLLISION TRIANGLE.
Collision Triangle
Inverse Collision Triangle at A2
Miss-Distance

- Look at points B1 and B2

- $V_R < 0$ just prior to these points and $V_R > 0$ immediately after these points.

- This implies that $R$ attains a minimum at the points B1 and B2.

- Hence, the miss-distance or the distance of closest approach occurs at these points.
\[ V_R^2 + V_{\theta}^2 = c^2 \]

\[ R\dot{V}_R = V_{\theta}^2. \]

\[ V_R^2 + R\dot{V}_R = c^2 \quad \Rightarrow \quad \dot{R}^2 + R\ddot{R} = c^2 \]

\[ R\ddot{R} = c^2 t + b \]

\[ c = \sqrt{V_{R0}^2 + V_{\theta0}^2} \]

\[ b = R_0 V_{R0} \]
Time at miss-distance

\[ 0 = c^2 t_{\text{miss}} + b \]

\[ \Rightarrow t_{\text{miss}} = \frac{-b}{c^2} \]

\[ \Rightarrow t_{\text{miss}} = \frac{-R_0 V_{R0}}{V_{R0}^2 + V_{\theta0}^2} \]
Miss-distance formula

\[
\frac{1}{2} R^2 = \frac{c^2 t^2}{2} + bt + a
\]

\[
R = \frac{R_0^2}{2}
\]

\[
R_{\text{miss}} = \sqrt{\frac{c^2 t^2_{\text{miss}}}{2} + 2bt_{\text{miss}} + 2a}
\]

\[
= R_0 \sqrt{\frac{V_{\theta_0}^2}{V_{\theta_0}^2 + V_{R_0}^2}}
\]
Effect of Lethal Radius

\[
    R_{\text{miss}} \leq R_{\text{lethal}}
\]

\[
    \Rightarrow R_0 \sqrt{\frac{V_{\theta_0}^2}{V_{\theta_0}^2 + V_{R_0}^2}} \leq R_{\text{lethal}}
\]

\[
    \Rightarrow R_0^2 V_{\theta_0}^2 \leq R_{\text{lethal}}^2 \left( V_{\theta_0}^2 + V_{R_0}^2 \right)
\]

\[
    \Rightarrow (R_0^2 - R_{\text{lethal}}^2) V_{\theta_0}^2 \leq R_{\text{lethal}}^2 V_{R_0}^2
\]

\[
    \Rightarrow V_{\theta_0}^2 \leq \left( \frac{R_{\text{lethal}}^2}{R_0^2 - R_{\text{lethal}}^2} \right) V_{R_0}^2
\]

\[
    \Rightarrow |V_{\theta_0}| \leq \sqrt{\frac{R_{\text{lethal}}^2}{R_0^2 - R_{\text{lethal}}^2}} |V_{R_0}|
\]

Capturability condition
Capture region

Figure 4.5: Capture region in the \((V_\theta, V_R)\)-space (a) Zero miss-distance (b) Non-zero miss-distance
Capture region formula

\[ V_{\theta_0} = 0 \]
\[ V_{R0} < 0 \]

\[ |V_{\theta_0}| \leq \eta |V_{R0}| \]
\[ V_{R0} < 0 \]

\[ \eta = \sqrt{\frac{R_{\text{lethal}}^2}{R_0^2 - R_{\text{lethal}}^2}} \]

Zero miss-distance

Non-zero miss-distance
Asymptotic Behaviour

\[ \begin{align*}
RV_R &= c^2 t + b \\
\Rightarrow V_R &= \frac{c^2 t + b}{R} \\
\Rightarrow V_R &= \frac{c^2 t + b}{\sqrt{c^2 t^2 + 2bt + 2a}} \\
V_\theta &= \sqrt{c^2 - V_R^2}
\end{align*} \]

\[ \begin{align*}
\lim_{t \to \infty} V_R &= c = \sqrt{V_{R0}^2 + V_{\theta0}^2} \\
\lim_{t \to \infty} V_\theta &= 0
\end{align*} \]
Figure 4.6: Variation of $V_\theta$ and $V_R$ against time
Avoidance Problems
Motivation

- Most work on aerospace obstacle avoidance model obstacles as circles (in 2-D) or spheres (in 3-D).
- For precision path planning and close proximity operations this is an overly conservative model, especially when we are dealing with small agile MAVs or quadrotors.
Inspiration

- Missile guidance

- Interception and avoidance are two sides of the same coin

- Interception kinematics modified to obtain collision conditions in the 2-D relative velocity space
Collision Cone

The **collision cone** is the collection of all velocity vector directions of an object A, moving with a given constant speed, for which a collision occurs with another object B moving with a fixed velocity.

**Collision Cone in 2-D between a point object and a circle**
Theorem 1: If a point and a circle are moving with constant velocities such that their initial conditions satisfy

\[ r_0^2 V_{\theta 0}^2 \leq R^2 (V_{r0}^2 + V_{\theta 0}^2) \text{ and } V_{r0} < 0 \quad (22) \]

then they are headed for a collision. The above conditions are both necessary and sufficient for a collision to occur.

R is the radius of the circle
**Point and Irregular Object**

![Diagram](image)

**Fig. 7.** Collision geometry between a point and an irregularly shaped object (a) \( \psi < \pi \) and (b) \( \psi > \pi \).

**Lemma 7:** If \( \psi < \pi \), \( A \) is on a collision course with \( B \) if and only if \( A \) is on a collision course with \( F \).

**Lemma 8:** If \( \psi > \pi \), then \( A \) is on a collision course with \( B \), if and only if \( A \) is not on a collision course with \( F^c \), and \( \vec{V}_A \neq \vec{V}_B \).
Collision Between Two Circles

Fig. 8. Collision geometry between two circles.
Lemma 9: If $\psi < \pi$, $A$ is on a collision course with $B$, if and only if $\mathcal{F}_1$ is on a collision course with $\mathcal{F}_2$. 
Lemma 10: If \( \pi < \psi < 2\pi \), \( A \) is on a collision course with \( B \), if and only if \( F_1 \) is not on a collision course with \( F_2 \), and \( \vec{V}_A \neq \vec{V}_B \).
Obstacle Avoidance

If a vehicle is headed for a collision with some object in its environment (that is, the robot’s velocity vector lies inside the collision cone), it can adopt any of the following three strategies to avert collision or to drive its velocity vector outside the collision cone.

1) Change speed (speed up, slow down, or reverse) but maintain heading direction. (Longitudinal acceleration)

2) Change heading direction (turn away from the original path) but keep speed constant. (Lateral acceleration)

3) Change both speed and heading direction. (Both longitudinal and lateral acceleration)
Obstacle Avoidance in a Dynamic Environment: A Collision Cone Approach

Animesh Chakravarthy and Debasis Ghose

Abstract—A novel collision cone approach is proposed as an aid to collision detection and avoidance between irregularly shaped moving objects with unknown trajectories. It is shown that the collision cone can be effectively used to determine whether collision between a robot and an obstacle (both moving in a dynamic environment) is imminent. No restrictions are placed on the shapes of either the robot or the obstacle, i.e., they can both be of any arbitrary shape. The collision cone concept is developed in a phased manner starting from existing analytical results—available in aerospace literature—that enable prediction of collision between two moving point objects. These results are extended to predict collision between a point and a circular object, between a point and an irregularly shaped object, between two circular objects, and finally between two irregularly shaped objects. Using the collision cone approach, several strategies that the robot can follow in order to avoid collision are presented. A discussion on how the shapes of the robot and obstacles can be approximated in order to reduce computational burden is also presented. A number of examples are given to illustrate both collision prediction and avoidance strategies of the robot.

Index Terms—Collision cone, dynamic environments, obstacle avoidance, path planning.

1. INTRODUCTION

OBSTACLE avoidance is a fundamental requirement in motion planning of a mobile robot. Several papers addressing this issue have appeared in robotics literature [6], [8], [13], [23], [24]. Motion planning can be categorized [6] as static (when the obstacles are stationary in the environment) or dynamic (when the obstacles are capable of movement and may even change shape and size). The environment could be completely known (when the trajectory of the obstacles is known a priori) or partially known (when obstacle trajectory is unknown or information about it is incomplete). This classification is not universal and an alternative classification is available in [8]. To date, a major research effort in this area has been applied to analyze and solve the problem of motion planning in a completely known environment with largely static and, to some extent, moving obstacles [23]. Their primary goal was to determine a collision-free path from a starting point to a goal point while optimizing some performance criterion. Configuration space approach, Voronoi diagrams, retraction methods, potential functions, visibility graphs, accessibility graphs, tangent graphs, etc. ([5], [8], [16]) are some of the techniques which have been reasonably successful in achieving this goal. While these approaches are justifiable for a completely known environment, a partially known dynamic environment—which is a more realistic framework in situations where obstacle motion cannot be predicted—requires a different approach. In fact, dynamic motion planning is more difficult than static motion planning even when complete information about the environment is available. This is shown by several available complexity results for motion planning [21].

Recent advances in robotics technology have made possible the development of autonomous and semi-autonomous robotic systems for land, air, and underwater operations. These robots use sophisticated onboard sensors to perceive their environment and use this information to plan and execute tasks [2], [3], [22], [23], [29]. Their primary use is in uncertain environments characterized by the presence of moving obstacles with unpredictable trajectories. Motion planning of robots in uncertain and unpredictable environments has attracted the attention of robotics researchers only recently [1], [7], [9]-[12], [19], [20], [25], [28].

In this paper, we present a novel approach called the collision cone approach which is ideally suited for automated guided vehicles or autonomous mobile robots. The method is new in the sense that it uses concepts which have their roots in aerospace literature rather than in robotics. The only relevant paper in the robotics literature that uses a similar concept is that by Tchonievich et al. [27].

The specific problem considered in this paper is that of a mobile robot avoiding one or more moving obstacles with unknown trajectories, based on sensor information collected by the robot. The robot and the obstacles are both assumed to move only by translation in a two-dimensional (2-D) space. Unlike previous literature, no assumptions are made on the shape of the robot or obstacles (i.e., they need not be polygonal—convex, or otherwise). They can be of any arbitrary shape but with the constraint that each is a single rigid body without relative motion between points on the body. Thus, this approach is more suitable for the problem of obstacle avoidance of an automated guided vehicle or a mobile robot in a workspace consisting of moving obstacles, rather than for motion planning of robotic manipulators.

This paper is motivated by the conviction that collision avoidance and collision achievement are, in principle, two aspects of the same problem. In the robotics literature, the problem of collision avoidance has attracted a considerable amount of attention. On the other hand, collision achievement
Interception Geometry in 3-D Point and Sphere

Engagement geometry between two point objects
Trajectories in \((V_\theta, V_\phi, V_r)\) Space
Point-Sphere Collision Cone

In an engagement between a point A and a sphere B (of radius R), collision occurs, if \( r_m \leq R \).

Thus, equation of the collision cone between a point and a sphere is given by:

\[
r_0^2 (V_{\theta 0}^2 + V_{\varphi 0}^2) \leq R^2 (V_{\theta 0}^2 + V_{\varphi 0}^2 + V_{r 0}^2); \quad V_{r 0} < 0
\]

Define a quantity \( y(t) = r_0^2 (V_{\theta 0}^2 + V_{\varphi 0}^2) - R^2 (V_{\theta 0}^2 + V_{\varphi 0}^2 + V_{r 0}^2) \)

Then, \( y(t) \leq 0; \quad V_{r 0} < 0 \) indicates that the heading of A lies inside the collision cone.

For constant velocities, we can show that \( dy/dt = 0 \).

For collision avoidance, A needs to perform a maneuver that will bring its velocity vector outside the collision cone (i.e. make \( y = 0 \)).

We can design avoidance maneuvers based on this idea.
Obstacle Avoidance in 3-D

- Thus, spherical bounding boxes for obstacles can be used.
- Note that in 2-D, even for arbitrary objects, the exact collision cone could be obtained by reducing the problem to obstacle avoidance between a point and a circle.
- However, unlike in 2-D, the exact 3-D collision cones can be non-convex.
- The tightest convex cone would be one which corresponds to the convex hull of the arbitrary obstacle.
- But this approach may not have the analyticity required to devise avoidance strategies.
Collision Avoidance of Elongated and/or Non-Convex Objects

- For such objects, a spherical approximation can become very conservative.
- We can achieve a less conservative approximation by using bounding quadric surfaces such as ellipsoids and hyperboloids.
Collision Cone of a Spheroid

A spheroid can be obtained by rotating an ellipse about its major axis.

Equation of a spheroid: \( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{b^2} = 1 \)

Points on the spheroid satisfy: \( r_1 + r_2 = 2a \), where \( a \) is the semi-major axis.

We can define two Lines-of-Sight \( AC_1 \) and \( AC_2 \), to the two foci of the spheroid.
Collision Cone of a Spheroid

Can integrate the ODE trajectories to show that:

\[ r_1^2 + r_2^2 = r_{m1}^2 + r_{m2}^2 + (V_{\theta 01}^2 + V_{\varphi 01}^2 + V_{r01}^2)(t_{m1} - t)^2 
+ (V_{\theta 02}^2 + V_{\varphi 02}^2 + V_{r02}^2)(t_{m2} - t)^2 \]

\[ r_m = A_1^2(1 + \tau V_1^2) + A_2^2(1 + \tau V_2^2) + 2A_1A_2\sqrt{1 + \tau(V_1^2 + V_2^2) + \tau V_1^2 V_2^2} \]

Where \( A_1, A_2, V_1, V_2, U_1, U_2 \) and \( \tau \) are functions of the relative velocity components of the point object to the two foci.

Thus, the expression for the collision cone is given by

\[ A_1^2(1 + \tau V_1^2) + A_2^2(1 + \tau V_2^2) + 2A_1A_2\sqrt{1 + \tau(V_1^2 + V_2^2) + \tau V_1^2 V_2^2} \leq 4a^2; \]

\[ V_{r02}U_1 + V_{r01}U_2 \leq 0 \]
Point and Irregular Object
Point and Irregular Object
Two Irregular Objects
Collision Cones in Closed Environments

Enclosing Space
Choosing a Path between Two Collision Cones: A Case of Multiple Obstacles

Length of the collision cone determines the time at which the collision may occur. So shorter collision cones are more risky than longer ones.


Collison Cones for Quadric Surfaces

Amitesh Chakravarthy, Member, IEEE, and Debasis Ghose, Member, IEEE

Abstract—The problem of collision prediction in dynamic environments appears in several diverse fields, which include robotics, air vehicles, underwater vehicles, and computer animation. In this paper, collision prediction of objects that move in 3-D environments is considered. Most work on collision prediction assumes objects to be modeled as spheres. However, there are many instances of object shapes where an ellipsoid or a hyperboloid-like boundary is more appropriate. In this paper, a collision cone approach is used to determine collision between objects whose shapes can be modeled by general quadric surfaces. Exact collision cones for such quadric surfaces are obtained in the form of algebraic expressions in the relative velocity space. For objects of arbitrary shapes, exact representations of planar sections of the 3-D collision cone are obtained.

Index Terms—Collision cone, dynamic environments, obstacle avoidance, path planning.

I. INTRODUCTION

Conditions that give rise to collision between two objects that move in space have been studied for researchers for decades. Current interest in microsatellite vehicles that operate in indoor environments or in low altitude flights through uneven terrain and vegetation, requires development of algorithms that suit these scenarios. This is the context in which this paper makes a contribution. In particular, it presents fairly general collision conditions for objects of different sizes and shapes that move in 3-D space.

The earlier works on collision conditions between point objects came from the missile guidance literature [1] and were used to develop empirical guidance laws. However, due to the point mass object assumption, these results had limited applications and did not satisfy the requirements of the robotics community, which was concerned with motions of objects with physical dimensions that operate in close proximity. Subsequently, two independently developed approaches of velocity obstacles [2] and collision cones [3] appeared. Both considered rectilinear motion of objects (say, A and B) on a plane and obtained conditions under which a collision occurs.

The velocity obstacles method [2] defined a set of points in the plane configuration space of A, such that if any velocity vector of object A (in terms of both magnitude and direction) lies in this set then collision occurs. This set is called a velocity obstacle. In later papers, [4], [5], the method was extended to objects with curved paths. However, it is not considered generalization to arbitrarily shaped objects.

The collision cone approach [3] on the other hand, considered the velocity of one of the objects (say, B) to be fixed and determined the cone of directions of the velocity of object A (with only its magnitude fixed) for which it collides with object B. This cone was termed the collision cone, and was used to generalize and obtain collision conditions even when the objects were of arbitrary shapes. This approach also yielded a closed form representation of the collision conditions. The assumption of fixed velocities in the collision cone approach makes it ideally suited for microsatellite vehicles which have limited control over longitudinal speed. Basic results on collision between points were used to obtain exact algebraic expressions for collision between irregularly shaped objects that move in a 2-D plane [3].

The collision cone method that is introduced in [3] has been used extensively in several papers, for example, [6]–[16]. In [6]–[8], the collision cone method was used to develop a driver assistance system for car collision avoidance. A sliding-mode controller was designed that used the collision cone computations as a reference input to generate the appropriate steering maneuver to guide the car out of an impending collision. Experimental results with a scaled remote-controlled car were also presented. In [9], the collision cone approach [3] and the potential field approach were combined to guide a mobile robot. Essentially, equations that govern the collision cone were used to determine a time-varying harmonic potential function [19], which was then used to guide a mobile robot toward its goal, while circumventing the obstacles. In [10]–[15], the collision cone concept was used for aircraft collision avoidance, which led to [16] and multivehicle collision avoidance.

Collision avoidance between two objects in 3-D space, which is of greater practical importance due to the inherent 3-D nature of collision avoidance problems in operations of MAVs, is not straightforward, except perhaps in the case of spheres. In fact, the papers cited earlier, which use the collision-cone concept in 3-D (for instance, [10]–[15]), as also other papers that discuss collision in 3-D (such as [18]–[19]), use the basic 2-D results and assume the objects to be spherical. The collision cone concept [3] used the relative velocity framework, from the aerospace guidance literature [11], [20], [21]. The contributions of this paper are the algebraic (closed form) representations of collision cones for 3-D objects represented by quadric surfaces. This representation is in the form of only two (as the case of a sphere or an ellipsoid) or three (as the case of a noncones. For example, linear, parabolic, or hyperbolic) inequalities. Thus, while collision conditions for objects are typically viewed in terms of conditions on the positions of points, the conditions derived in this paper comprise the same into conditions that involve only two points. These inequalities represent exact collision conditions when objects move with constant velocities and no rotations. For varying (and a priori unknown) velocities, these results can still be applied in an approximate sense by assumption of the obstacle velocity to be piecewise-constant in time (through use of incremental sensor measurements, for example) and constructing collision cones that are approximately piecewise-constant in time. These can be used for both collision prediction as well as determination of the magnitude of maneuvers required for collision avoidance. This paper generalizes the results in [3] to 3-D.

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1. Introduction

An important component of robot path planning is the obstacle avoidance problem, that is, determining a safe trajectory of a robotic vehicle so that it circumvents various stationary as well as dynamic obstacles of arbitrary shapes. A commonly used metric for many path planning algorithms is obstacle complexity, or the amount of information used to store a computer model of the obstacle [Kim et al. 2006]. Obstacle complexity can be measured in terms of the number of obstacle edges (Goerzen et al. 2010). A common practice then is to use a bounding box approximation for the objects. The obstacle avoidance conditions are then computed for the bounding box as a whole—this is as opposed to performing a spatial discretization of the object and then computing avoidance conditions for each such discrete component separately. In particular, generation of analytical expressions that give collision avoidance conditions for such bounding boxes are particularly helpful, since they can lead to tremendous computational savings, especially in multi-obstacle environments.

The choice of a bounding box involves the inherent tradeoff that exists between computational ease and accuracy/precision and also must account for the dimension of the physical space. In 2-D case, even when the object is of arbitrary shape, a circular bounding box has both perfect precision/accuracy as well as analytical form Chakravarthy and Ghose (1998). In 3-D this no longer holds and a straight-forward extension to spherical bounding surface can only be analytical but it cannot satisfy the requirement of precision to a satisfactory level. This is especially true when objects are expected to move in close proximity to each other. Thus, while spheres do serve as common bounding box approximations for many objects in space (because of the computational ease associated with them), these approximations
Quadrotor Obstacle Avoidance

Circular bounding box for the quadrotor has more difficulty in passing through a gap than the elliptical bounding box.
Elliptical bounding box demands less control magnitude than the circular bounding box for the obstacle.
Quadrotor Trajectory Avoiding Two Elliptical Obstacles
End of Lecture 2     THANK YOU