From Finite Elements to a Virtual Experimentation Station for damaging composites

C.S. UPADHYAY

and

V. MURARI, R. DHAMA

DEPARTMENT OF AEROSPACE ENGINEERING
INDIAN INSTITUTE OF TECHNOLOGY, KANPUR
Structure Analysis Lab and LAAMS
Department of Aerospace Engineering, IIT Kanpur
MOTIVATION

• WHY DO WE COMPUTE?

• CAN EFFECTIVE ANALYSIS COUPLED WITH GOOD MODELING REDUCE EXPERIMENTATION?

• CAN WE IMPROVE STRUCTURAL DESIGN BASED ON COMPUTATIONAL ANALYSIS?

- VIRTUAL EXPERIMENTATION STATION
- Need to create samples and test – DCB, tensile.
- Create models based on observations.
- Conduct standardized tests on crack density, damage growth
- Check model against actual specimen level tests (e.g. laminated plate with a hole)
- Models with rate dependence (viscoplasticity type)
- Effect of process, residual state?
- Quality of cure – porosity, wetting, etc. How to model these?

CHALLENGES IN COMPOSITE BLADE DESIGN

DELAMINATION

Laminate behavior can be described using elementary entities called meso-constituents (the ply and the interface).

The response of a damaged layer, at any instant of time (load-state), can be expressed in terms of degradation in elastic moduli and in-elastic strains due to damage and/or matrix plasticity.
To develop a damage model for unidirectional thermosetting matrix composites with the following features.

1. **Simple**
   - Simple to obtain the model parameters.

2. **Physically Based**
   - Internal or damage variables, Evolution laws, etc - based on the physical observations (thought/constituent level numerical experiments)

3. **Thermodynamically Consistent**
   - The dissipation of energy is always positive for irreversible process (the damage never heals by itself)

4. **Accounting for the Multiple Scales**
   - The existence and influences of different length scales
Cross-sectional view of [0/±45/90]s carbon/epoxy laminate
Dostal CA (1987), Engineered Materials Handbook (Composites), ASM International
DEFINITIONS:

TRANSVERSE CRACK

LOCAL DEBOND

DIFFUSE DAMAGE MECHANISMS

(a) Matrix Breakage
(b) Fiber Breakage
(c) Fiber Matrix De-bonding
(d) Matrix Cracks
(e) Fiber Splitting
**Damage Mechanisms**

### Micro-mechanisms
- **Fiber break**
  - Local Normal stress, \( \sigma_{11} \)
- **Matrix cracks**
  - Local Shear stress, \( \sigma_{12}, \sigma_{13}, \sigma_{23} \); Local Transverse stress, \( \sigma_{22}, \sigma_{33} \)
- **Fiber debond**
  - Local Shear stress, \( \sigma_{12}, \sigma_{13}, \sigma_{23} \); Local Transverse stress, \( \sigma_{22}, \sigma_{33} \)
  - (occurs when fiber-matrix interface is weak)

### Meso-mechanisms
- **Transverse Cracks**
  - Matrix damage growth to cover full thickness followed by growth along fiber direction.
- **Delamination**
  - Damage at the ply interface
**Mathematical theory of Homogenization**

**GOVERNING EQUATION:**

\[-\frac{\partial}{\partial x_j} \left[ C_{ijkl}^\varepsilon \epsilon_{kl} (u^\varepsilon) \right] = f^\varepsilon\]

\[C_{ijkl}^\varepsilon (x) = \begin{cases} 
C_{ijkl}^f, & \text{IF } x \text{ IS ON FIBER} \\
C_{ijkl}^m, & \text{IF } x \text{ IS ON MATRIX} 
\end{cases}\]

**LENGTH SCALES**

\[\varepsilon \ll L\]

**Effective properties can be assigned**
SEVERAL FORMULATIONS EXIST

PERIODIC SOLUTION IS MATHEMATICALLY RIGOROUS

\[ \left( \frac{\partial}{\partial y_j} C_{ijkl} e_{kl}^y (\chi^{rs}) \right) = \left( \frac{\partial}{\partial y_j} (C_{ijrs} \delta_{rs}) \right) \]

Function $\chi$ is $y$-periodic
Homogenization Formulation

Relation between the macro and the micro strain through strain concentration matrix

\[ e_{kl}(u^\varepsilon) = e_{kl}^x(u) + e_{kl}^y(-\chi^{rs}(y)e_{rs}^x(u)) = M_{k\ell rs}e_{rs}^x(u) \]

Strain Concentration Factor Matrix

\[ M_{k\ell rs} = \begin{bmatrix} \delta_{rk} \delta_{sl} - e_{kl}^y(\chi^{rs}(y)) \end{bmatrix} \]

Effective Properties

\[ C_{ij\ell rs}^{avg} = \frac{1}{V_{\text{RVE}}^2} \int C_{ijkl}M_{k\ell rs}dV \]

Effective coefficient of thermal expansion

\[ \bar{\alpha}_{rs} = \left[ C_{ij\ell rs} \right]^{-1} \beta_{ij} \]

\[ \beta_{ij} = \left\langle C_{ijkl}^\varepsilon \alpha_{kl}^\varepsilon \right\rangle = C_{ijkl}^f\alpha_{kl}^f\nu_f + C_{ijkl}^m\alpha_{kl}^m\nu_m \]
Linear Tetrahedral Elements are used. A commercial software, HYPERMESH, is used only for meshing.
## Code Validation

<table>
<thead>
<tr>
<th>Elas. Const.</th>
<th>Code</th>
<th>References at $V_f = 0.47$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$V_f = 0.407$</td>
<td>$V_f = 0.503$</td>
</tr>
<tr>
<td>$E_{11}$ (GPa)</td>
<td>195.0</td>
<td>225.0</td>
</tr>
<tr>
<td>$E_{22}$ (GPa)</td>
<td>129.0</td>
<td>152.0</td>
</tr>
<tr>
<td>$E_{33}$ (GPa)</td>
<td>129.0</td>
<td>152.0</td>
</tr>
<tr>
<td>$G_{23}$ (GPa)</td>
<td>42.2</td>
<td>48.2</td>
</tr>
<tr>
<td>$G_{13}$ (GPa)</td>
<td>48.9</td>
<td>57.7</td>
</tr>
<tr>
<td>$G_{12}$ (GPa)</td>
<td>48.9</td>
<td>57.7</td>
</tr>
<tr>
<td>$\nu_{23}$</td>
<td>0.27</td>
<td>0.24</td>
</tr>
<tr>
<td>$\nu_{13}$</td>
<td>0.20</td>
<td>0.18</td>
</tr>
<tr>
<td>$\nu_{12}$</td>
<td>0.20</td>
<td>0.18</td>
</tr>
</tbody>
</table>
Stress profile in undamaged RVE
DAMAGE $\Rightarrow$

$E_{\text{damaged}} = E_{\text{undamaged}} (1-d)$

WHY?
**Micromechanical Analysis**

<table>
<thead>
<tr>
<th>Parameters considered for micromechanical analysis</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Value 3</th>
<th>Value 4</th>
<th>Value 5</th>
<th>Value 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$v_f$</td>
<td>0.246</td>
<td>0.302</td>
<td>0.407</td>
<td>0.503</td>
<td>0.608</td>
<td>0.636</td>
</tr>
<tr>
<td>$d_1$</td>
<td>0.0</td>
<td>1.0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$d_2$</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.8</td>
</tr>
<tr>
<td>$d_3$</td>
<td>0.0</td>
<td>0.2</td>
<td>0.4</td>
<td>0.5</td>
<td>0.6</td>
<td>0.8</td>
</tr>
</tbody>
</table>

**Effect of different damage modes on effective properties**

- Fiber breakage
- Fiber-debond
- Matrix cracks
Material Degradation Model

Effective elastic property can be defined as, \( E = E \left( v_f, d_i \right), \quad i = 1, 2, ... ndam \)

Reference Configuration: Undamaged RVE with volume fraction=0.503

\[
E \left( 0.503 + \Delta v_f, 0.0 + \Delta d_i \right) \approx E \left( 0.503, 0.0 \right) + \\
\frac{\partial E}{\partial v_f} \Delta v_f \left|_{(0.503,0.0)} \right. + \sum_{i=1}^{3} \left\{ \frac{\partial E}{\partial d_i} \Delta d_i \right|_{(0.503,0.0)} \right\} + \\
\left[ \frac{\partial^2 E}{\partial v_f^2} \Delta v_f^2 \right|_{(0.503,0.0)} + \sum_{i=1}^{3} \left\{ \frac{\partial^2 E}{\partial d_i^2} \Delta d_i^2 \right|_{(0.503,0.0)} \right\} + \\
+ \sum_{i=1}^{3} \left\{ 2 \frac{\partial^2 E}{\partial v_f \partial d_i} \Delta v_f \Delta d_i \right|_{(0.503,0.0)} \right\}
\]

\[
\Delta E \approx [\alpha_1 \Delta v_f + \alpha_2 \Delta v_f^2] + \sum_{i=1}^{3} [\beta_1 i \Delta d_i + \beta_2 i \Delta d_i^2] + \sum_{i=1}^{3} [\chi_i \Delta v_f \Delta d_i]
\]
Fiber breakage
- not only $E_{11}$
- also $G_{12}$, $G_{13}$, $\nu_{12}$, $\nu_{13}$

\[
E_d = \frac{1}{2 (1 - d_F)} \left[ \frac{\left\langle \bar{\sigma}_{11} \right\rangle^2}{E_1} + \phi \left( \frac{\left\langle -\bar{\sigma}_{11} \right\rangle^2}{E_1} \right) - \left( \frac{\nu_{21}^0}{E_2} + \frac{\nu_{12}^0}{E_1} \right) \bar{\sigma}_{11} \bar{\sigma}_{22} - \left( \frac{\nu_{31}^0}{E_3} + \frac{\nu_{13}^0}{E_1} \right) \bar{\sigma}_{11} \bar{\sigma}_{33} \right. \\
- \left( \frac{\nu_{32}^0}{E_3} + \frac{\nu_{23}^0}{E_2} \right) \bar{\sigma}_{22} \bar{\sigma}_{33} \right] + \\
\frac{1}{2 (1 - d')} \left[ \frac{\left\langle \bar{\sigma}_{22} \right\rangle^2}{E_2} + \frac{\left\langle \bar{\sigma}_{33} \right\rangle^2}{E_3} \right] + \\
\frac{1}{2 (1 - d)} \left[ \frac{\bar{\sigma}_{12}^2}{G_{12}} + \frac{\bar{\sigma}_{23}^2}{G_{23}} + \frac{\bar{\sigma}_{31}^2}{G_{31}} \right] + \\
\frac{1}{2} \left[ \frac{\left\langle -\bar{\sigma}_{22} \right\rangle^2}{E_2} + \frac{\left\langle -\bar{\sigma}_{33} \right\rangle^2}{E_3} \right]
\]
Fiber-matrix debond
- not only $G_{23}$, $G_{13}$, $G_{12}$
- also $E_{22}$, $E_{33}$

$$E_d = \frac{1}{2 (1 - d_F)} \left[ \frac{\langle \sigma_{11} \rangle^2}{E_1^0} + \frac{\phi \langle -\sigma_{11} \rangle^2}{E_1^0} - \left( \frac{\nu_{21}^0}{E_2^0} + \frac{\nu_{12}^0}{E_1^0} \right) \bar{\sigma}_{11} \bar{\sigma}_{22} - \left( \frac{\nu_{31}^0}{E_3^0} + \frac{\nu_{13}^0}{E_1^0} \right) \bar{\sigma}_{11} \bar{\sigma}_{33} \right]$$

$$- \left( \frac{\nu_{32}^0}{E_3^0} + \frac{\nu_{23}^0}{E_2^0} \right) \bar{\sigma}_{22} \bar{\sigma}_{33} \right] +$$

$$\frac{1}{2 (1 - d')} \left[ \frac{\langle \sigma_{22} \rangle^2}{E_2^0} + \frac{\langle \sigma_{33} \rangle^2}{E_3^0} \right]$$

$$\frac{1}{2 (1 - d)} \left[ \frac{\sigma_{12}^2}{G_{12}^0} + \frac{\sigma_{23}^2}{G_{23}^0} + \frac{\sigma_{31}^2}{G_{31}^0} \right] +$$

$$\frac{1}{2} \left[ \frac{\langle -\sigma_{22} \rangle^2}{E_2^0} + \frac{\langle -\sigma_{33} \rangle^2}{E_3^0} \right]$$
Residual Thermal Stress

- Cooling down of completely cured sample from the cure temperature

Stress:
- Radial

Material:
- Glass/epoxy

Volume Fraction:
- 0.503

$\Delta T = 80^\circ C$
Yield strength of matrix (Epoxy): 58 MPa
Eff. Coefficient of Thermal Expansion (CTE)

Damage: matrix cracks
Material: glass/epoxy
Volume Fraction: 0.503
ΔT = 80°C
**FREE ENERGY FUNCTION**

**DEFINITIONS**

\[
E_{ij}^D = \begin{cases} 
E_{ij}^D(d_1, d_2, d_3), & \text{when } \sigma_{11}, \sigma_{22} \text{ or } \sigma_{33} > 0 \\
E_{ij}^D(0, d_2, d_3), & \text{when } \sigma_{11} < 0, \sigma_{22} \text{ or } \sigma_{33} > 0 \\
E_{ij}^D(d_1, d_2, 0), & \text{when } \sigma_{11} > 0, \sigma_{22} \text{ or } \sigma_{33} < 0 \\
E_{ij}^D(0, d_2, 0), & \text{when } \sigma_{11}, \sigma_{22} \text{ and } \sigma_{33} < 0
\end{cases}
\]

- \(d_1\) Fiber breakage
- \(d_2\) Fiber matrix debond
- \(d_3\) Matrix cracks

**PROPOSED FREE ENERGY FUNCTION FOR A DAMAGED COMPOSITE**

\[
U^D = \frac{1}{2} \left\{ \frac{\sigma_{11}^2}{E_{11}} + \frac{\sigma_{22}^2}{E_{22}} + \frac{\sigma_{33}^2}{E_{33}} + \\
- \left( \frac{\nu_{21}^D}{E_{22}} + \frac{\nu_{12}^D}{E_{11}} \right) \sigma_{11}\sigma_{22} - \left( \frac{\nu_{31}^D}{E_{33}} + \frac{\nu_{13}^D}{E_{11}} \right) \sigma_{11}\sigma_{33} - \\
- \left( \frac{\nu_{32}^D}{E_{33}} + \frac{\nu_{23}^D}{E_{22}} \right) \sigma_{22}\sigma_{33} + \frac{2\sigma_{12}^2}{G_{12}^D} + \frac{2\sigma_{13}^2}{G_{13}^D} + \frac{2\sigma_{23}^2}{G_{23}^D} \right\}
\]
Initiation criteria - Critical Regions

\( \sigma_{11}^{\text{mac}} = 0.207 \sigma_x, \quad \sigma_{22}^{\text{mac}} = 0.793 \sigma_x, \quad \sigma_{12}^{\text{mac}} = 0.414 \sigma_x \)
Micromechanical Analysis
(To understand the damage modes and sites of failure initiation)

Mises stress distributions

Stress Concentrations

- $\sigma_{11}^{\text{matrix}} \approx 0.03\sigma_{11}^{\text{meso}}$
  (hardly any concentration in matrix)

- $\sigma_{12}^{\text{matrix}} \approx 1.6\sigma_{12}^{\text{meso}}$
  Concentration $\rightarrow$ where fibers are closest

- $\sigma_{22}^{\text{matrix}} \approx 1.6\sigma_{22}^{\text{meso}}$
  Concentration $\rightarrow$ where fibers are closest

- For $\sigma_{23}^{\text{matrix}}$
  Concentration $\rightarrow$ around the fiber
Damage Initiation

- **Fiber Break:** \( d_1 = 0 \) while, \( \sigma_{11} \leq \Lambda \sigma_{11}^f \)

- Fiber-matrix debond \( (d_2) \) and matrix cracks \( (d_3) \) are assumed to be matrix phenomenon.

- The critical regions are identified in the matrix.

- **Von-Mises criteria** in the matrix is used for obtaining the initiation criteria.

\[
\left( \sigma_{y}^{mic} \right)^2 \approx \frac{1}{2} \left[ \left( \sigma_{11}^{mic} - \sigma_{22}^{mic} \right)^2 + \left( \sigma_{22}^{mic} - \sigma_{33}^{mic} \right)^2 + \left( \sigma_{11}^{mic} - \sigma_{33}^{mic} \right)^2 \right] + 3 \left( \sigma_{23}^{mic} \right)^2 + \left( \sigma_{31}^{mic} \right)^2 + \left( \sigma_{12}^{mic} \right)^2
\]

For Fiber-matrix debond, \( \sigma_{12}^{mic} \approx 3 \left( \sigma_{12}^{mic} \right)^2 \)

\[ \sigma_{12}^{mic} = 1.7257 \sigma_{12}^{mac} \]

\[ \sigma_{12}^{mac} = \frac{1}{1.7257 \sqrt{3}} \sigma_y \]

For Matrix cracks, \( \sigma_{22}^{mic} \approx \left( \sigma_{22}^{mic} \right)^2 \)

\[ \sigma_{22}^{mic} = 1.6949 \sigma_{22}^{mac} \]

\[ \sigma_{22}^{mac} = \frac{1}{1.6949} \sigma_y \]

Note:
- Initiation parameters obtained directly from constituent properties
- No lamina tests required
**Damage Evolution**

\[ d_1 = 0; \text{ when, } \sigma_{11} \leq a\sigma^f_{11} \]

\[ d_1 = \frac{\sigma_{11} - a\sigma^f_{11}}{(1-a)\sigma^f_{11}}; \text{ when, } \sigma_{11} > a\sigma^f_{11} \]

**Fiber breakage**

\[ d_2 = 0; \text{ when, } |\sigma_{12}| \leq |\sigma^c_{12}| \]

\[ d_2 = c_1 (|\sigma_{12}| - |\sigma^c_{12}|); \text{ when, } |\sigma_{12}| > |\sigma^c_{12}| \]

**Fiber matrix debond**

\[ d_3 = 0; \text{ when, } \sigma_{22} \leq \sigma^c_{22} \]

\[ d_3 = c_2 (\sigma_{22} - \sigma^c_{22}); \text{ when, } \sigma_{22} > \sigma^c_{22} \]

**Matrix cracks**
Inelastic Response – Matrix Plasticity

(a) only $\sigma_{12}^{\text{meso}} = 1$; all other $\sigma_{ij}^{\text{meso}} = 0$

(b) only $\sigma_{12}^{\text{meso}} = 1$; all other $\sigma_{ij}^{\text{meso}} = 0$

Experiment
Assumed model

Matrix $E_m$
Softened zone $0.01E_m$

$G_{12}^d = G_{12}^0$
$G_{12}^{y,d} = \omega G_{12}^d = \omega G_{12}^0$

$G_{12}^{y,d} = \omega (1 - d_2)^2 G_{12}^d$
Inelastic Response – Shift in Initial State

Difference in CTE and CCS between fiber and matrix $\Rightarrow$ Initial residual state of stress and strain

Damage initiation and evolution $\Rightarrow$ Change in initial residual state of stress and strain

Quantification of these changes $\Rightarrow$ Effect of damage on effective CTE and CCS

$$\bar{\alpha}_{rs} = \left[ \bar{S}_{rsij} \right] \left\{ C_{ijf}^{f} \alpha_{k}^{f} \bar{v}_f + C_{ijb}^{m} \alpha_{k}^{m} \bar{v}_m \right\} ; \quad e_{rs}^{T} = \bar{\alpha}_{rs} \Delta T;$$

$$\bar{\eta}_{rs} = \left[ \bar{S}_{rsij} \right] \left\{ C_{ijf}^{f} \eta_{k}^{f} \bar{v}_f + C_{ijb}^{m} \eta_{k}^{m} \bar{v}_m \right\} ; \quad e_{rs}^{S} = \bar{\eta}_{rs} \frac{\Delta V}{V};$$

$A_0$ - Residual strain
$B_0$ - Residual strain with damage
$A_1$ - Reference state for measurements
$L_1$ - Damage initiates
$L_1 - L_2$ - Loading
$L_1 - L_2$ - Damage grows
$L_1 - B_1$ - Unloading
$A_1 - B_1$ - Inelastic strains
Model Identification

\[ \sigma_{11} = 0.8753\sigma_x; \quad \sigma_{22} = 0.1247\sigma_x; \quad \sigma_{12} = -0.5\sigma_x; \]

Using \([\pm 45]_s\)

Lamina stresses

\[ d_2 = c_1 \left( |\sigma_{12}| - |\sigma_{12}^c| \right) \]

For different \(|\sigma_{12}| > |\sigma_{12}^c|\)

- Measure \(\Delta G_{12}\)
- Compute \(d_2\)

**Figure**: Cyclic lamina shear stress-strain response in \([\pm 45]_s\) T300/914 carbon/epoxy laminates (Ladeveze et al, 1992)
Model Identification

using \([\pm 67.5]_s\)

**lamina stresses**

\[ \sigma_{11} = 0.1148 \bar{\sigma}_x; \quad \sigma_{22} = 0.8852 \bar{\sigma}_x; \quad \sigma_{12} = -0.3219 \bar{\sigma}_x; \]

\[d_3 = c_2 (\sigma_{22} - \sigma_{22}^c)\]

For different \(\sigma_{22} > \sigma_{22}^c\)

- Measure \(\Delta E_{22}\)
- \(d_2\) evolution - known
- subtract \(\Delta E_{22} \big|_{d_2}\)
- compute \(d_3\) for \(\Delta E_{22} \big|_{d_3}\)

**Figure:** Transverse stress-strain response of lamina of \([\pm 67.5]_s\) T300/914 carbon/epoxy laminates (Ladeveze et al, 1992)
**Model Predictions**

- MICRO-MECHANICS BASED DETERMINATION OF EFFECTIVE PROPERTIES
- DETERMINATION OF REGIONS OF STRESS CONCENTRATION
- ACCURATE PREDICTION OF ONSET OF FAILURE
- EASY DETERMINATION OF GROWTH PARAMETERS (DESIGN OF EXPERIMENTS)
REFINEMENTS IN THE MODEL - MATRIX PLASTICITY
Matrix crack initiates after diffuse damage grows to a critical length, and then (is assumed to) propagates at the same rate as the diffuse damage – 2nd Expt. of Ladeveze?
**Coupling of diffuse damage with matrix crack**

**Table : Without diffuse damage**

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_{11}^{\text{macro}}$</th>
<th>$\sigma_{22}^{\text{macro}}$</th>
<th>$\sigma_{33}^{\text{macro}}$</th>
<th>$\sigma_{23}^{\text{macro}}$</th>
<th>$\sigma_{13}^{\text{macro}}$</th>
<th>$\sigma_{12}^{\text{macro}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{11}^{\text{micro}}$</td>
<td>0.0318</td>
<td>-0.1584</td>
<td>0.9220</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{22}^{\text{micro}}$</td>
<td>0.0020</td>
<td>-0.0685</td>
<td>0.8740</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{33}^{\text{micro}}$</td>
<td>-0.0038</td>
<td>-0.3451</td>
<td>1.7138</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{23}^{\text{micro}}$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.2905</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{13}^{\text{micro}}$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>1.7377</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{12}^{\text{micro}}$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0649</td>
</tr>
</tbody>
</table>

**Table : With diffuse damage ($d_2 = 0.2$)**

<table>
<thead>
<tr>
<th></th>
<th>$\sigma_{11}^{\text{macro}}$</th>
<th>$\sigma_{22}^{\text{macro}}$</th>
<th>$\sigma_{33}^{\text{macro}}$</th>
<th>$\sigma_{23}^{\text{macro}}$</th>
<th>$\sigma_{13}^{\text{macro}}$</th>
<th>$\sigma_{12}^{\text{macro}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_{11}^{\text{micro}}$</td>
<td>0.0726</td>
<td>0.0779</td>
<td>0.2661</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{22}^{\text{micro}}$</td>
<td>0.0195</td>
<td>0.3768</td>
<td>0.1297</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{33}^{\text{micro}}$</td>
<td>-0.0002</td>
<td>-0.0399</td>
<td>0.3685</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{23}^{\text{micro}}$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.5109</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{13}^{\text{micro}}$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.9777</td>
<td>0.0000</td>
</tr>
<tr>
<td>$\sigma_{12}^{\text{micro}}$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1073</td>
</tr>
</tbody>
</table>
COUPLED STIFFNESS REDUCTION MODEL

Stiffness reduction model

\[
E(0.503 + \Delta v_f, 0.0 + \Delta d_i) \approx E(0.503, 0.0) + \left. \frac{\partial E}{\partial v_f} \right|_{(0.503,0.0)} \Delta v_f + \left. \frac{\partial E}{\partial d_i} \right|_{(0.503,0.0)} \Delta d_i + \sum_{i=1}^{3} \left. \frac{\partial^2 E}{\partial d_i^2} \right|_{(0.503,0.0)} \Delta d_i^2 + \sum_{i=1}^{3} \left. \frac{\partial^2 E}{\partial v_f^2} \right|_{(0.503,0.0)} \Delta v_f^2 + \sum_{i=1}^{3} \left. \frac{\partial^2 E}{\partial d_i \partial d_j} \right|_{(0.0,0.0)} \Delta d_i \Delta d_j
\]

\[
j_1 = \left. \frac{\partial E}{\partial v_f} \right|_{(0.503,0.0)}, \quad j_2 = \frac{1}{2!} \left. \frac{\partial^2 E}{\partial v_f^2} \right|_{(0.503,0.0)}, \quad k_{1i} = \frac{1}{2!} \left. \frac{\partial E}{\partial d_i} \right|_{(0.503,0.0)}, \quad k_{2i} = \left. \frac{\partial^2 E}{\partial d_i^2} \right|_{(0.503,0.0)}
\]

\[
p_i = \left. \frac{\partial^2 E}{\partial v_f \partial d_i} \right|_{(0.503,0.0)}, \quad i = 1, 2, 3 \quad m_{ij} = \left. \frac{\partial^2 E}{\partial d_i \partial d_j} \right|_{(0.0,0.0)}, \quad i = 1, 2 \text{ and } j = i + 1
\]
Model formulation

- Dissipation due to damage
- Dissipation due to plastic matrix

\[
\langle \sigma_{ij} \rangle = (\nu_m - \nu_m^d) \langle \sigma_{ij}^{ud} \rangle + \nu_m^d \langle \sigma_{ij}^d \rangle + \nu_f \langle \sigma_{ij}^f \rangle
\]

where,

\[
\nu_m^d = \text{Volume fraction of plastic region of matrix}
\]

\[
\nu_m = \text{Total volume fraction of matrix}
\]

\[
\nu_f = \text{Volume fraction of fiber}
\]

\[
\sigma_{ij}^d = \text{Stress in plastic region of matrix}
\]

\[
\sigma_{ij}^{ud} = \text{Stress in elastic region of matrix}
\]

\[
\sigma_{ij}^f = \text{Stress in fiber}
\]
Model formulation continued......

- Using Constitutive relation in previous equation

\[
\langle \sigma_{ij} \rangle = (\nu_m - \nu_m^d) C_{ijkl}^m \langle \varepsilon_{kl}^{ud} \rangle + \nu_m^d C_{ijkl}^m \langle \varepsilon_{kl}^{e,d} \rangle + \nu_f C_{ijkl}^f \langle \varepsilon_{kl}^f \rangle
\]

\( \varepsilon_{kl}^{e,d} = \) Elastic strain in plastic region of matrix

\( \varepsilon_{kl}^{ud} = \) Total strain in elastic region of matrix

- The above equation is rearranged in the form given below

\[
\langle \sigma_{ij} \rangle = C_{ijkl}^d \left( \langle \varepsilon_{kl} \rangle - \nu_m^d (C_{ijkl}^d)^{-1} C_{ijkl}^m \langle \varepsilon_{ij}^R \rangle \right)
\]

\( \varepsilon_{ij}^R = \) Plastic strain at matrix failure

- Volume fraction of damaged region of matrix, \( \nu_m^d = d_2 \nu_m \), which implies

\[
\langle \varepsilon_{kl}^p \rangle = d_2 \nu_m (C_{ijkl}^d)^{-1} C_{ijkl}^m \langle \varepsilon_{ij}^R \rangle
\]
Proposed model

- Global plastic strain is given by

\[ \langle \varepsilon_{kl}^p \rangle = d_2 \nu_m \left( C_{ijkl}^d \right)^{-1} C_{ijkl}^m \langle \varepsilon_{ij}^R \rangle \]  

- Plastic strain is directly proportional to the volume fraction of matrix and size of damage

![Graphs showing plasticity evolution for carbon/epoxy and glass/epoxy](image)

**Figure:** Plasticity evolution for (a) carbon/epoxy (b) glass/epoxy
- Curve fitting of the form, $f = ae^{bd_2}$, can be used in the plastic model.

**Table: Evolution parameters**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0.6</td>
</tr>
<tr>
<td>$b$</td>
<td>0.9761</td>
</tr>
</tbody>
</table>

**Figure: Shear response of \([\pm 45^0]_s\) laminate [4]**

**Figure: Without curve fitting**

**Figure: With curve fitting**
MODEL PREDICTIONS

Assume some macro shear stress

\[ d_2 = 0.110 (\Gamma - 2.573) \]

\[ \downarrow \]

Find \( d_2 \) using damage evolution model

\[ \downarrow \]

Find \( \Delta E \) using stiffness reduction model

\[ \downarrow \]

Obtain \( \varepsilon^p \) using plasticity model and

\[ \varepsilon^e = \frac{\sigma_{12}}{2E} \]

\[ \downarrow \]

Plot stress vs. total strain
NATURAL FIBERS !!!

The Mormut- lovable Himalayan animal

Wild Yaks grazing around frozen lake

Pangong Lake