An Algebraic Geometric Approach to Analyze Static Voltage Collapse in a Simple Power System Model

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Abstract — This paper presents an algebraic geometric method to analyze static voltage collapse in a simple power system model. The method is based on computing a lexicographic (lex) ordered Gröbner basis for the ideal generated by the parameterized load-flow equations (LFEs). Computing the solutions to the LFEs is then equivalent to computing the algebraic variety corresponding to this ideal. Incidentally, this method allows the determination of all the solutions to the set of LFEs. The formulation also allows the determination of an explicit polynomial relation between the bus voltage components and the real/reactive loadings at that bus. Using this relation, we show that (i) the sensitivity of the bus voltage (to the real/reactive loadings) can be analytically expressed as rational functions and (ii) the $PV$, $QV$ curves up to the point of collapse can be determined without resorting to repeated load-flow calculations. The proposed approach is exemplified on a simple three bus power system along with a discussion on its limitations.

I. INTRODUCTION

The reliable functioning of a myriad basic entities in today’s complex world depends upon the secure operation of the electric power grid which is now considered a critical infrastructure, [1]. Among several factors that can threaten the security of a power system, the phenomenon of voltage (in)stability has received considerable attention in the recent years. Broadly speaking, voltage stability, as characterized by a gradual or sudden decline in voltage levels possibly culminating in a collapse [2], is observed following a suitable disturbance, typically, in stressed power systems. While there are both static and dynamic approaches to study voltage collapse [3] often, it is acceptable [2] to use the former means to simplify the analysis. One of the important goals in voltage collapse studies is to determine the maximal limit to which a bus in a power system can be subject to smooth changes in loading without a loss of equilibrium. Assuming the load flow equations (LFE) as the equilibrium model, a number of methods that exploit the properties of the LFEs, [5] and its corresponding Jacobian, [4], [5], have been proposed to determine this limit. In principle, tracing the solutions to the LFEs with smooth changes in the load up to the point when load-flow Jacobian becomes singular, is one method to determine this limit at which, the system is said to undergo a static collapse, [6]. Since conventional load-flow algorithms can suffer from convergence problems as the Jacobian becomes progressively ill-conditioned, the continuation based power flow (CPFLOW), [7],[8] was proposed to circumvent this issue. CPFLOW based methods use a predictor-corrector scheme to calculate solutions to parameterized LFEs while remaining well conditioned in the vicinity of the collapse point. However, the existing LFE based methods to determine the point of static collapse are based on numerical techniques. In this paper, we present an algebraic-geometry based method to analyze static voltage collapse in a simple power system model. While an earlier work [9] reports on the application of Gröbner basis for solving the load flow problem, it neither address the voltage collapse/sensitivity phenomenon nor correlates with relevant power systems literature besides ignoring the proper concepts from a load flow formulation.

Here, the LFEs are parameterized in terms of the bus loadings and a lexicographic (lex) ordered Gröbner basis is computed for the ideal generated by this set of load-flow equations (LFEs). In algebraic-geometric terms, the solutions to the LFEs are equivalent to computing the variety corresponding to this ideal. A useful consequence of the lex ordering is that the set of LFEs which have been transformed into a set of polynomial equations, the zeroes of which can be successively computed in a fashion similar to Gaussian elimination method to arrive at the load-flow solutions. Incidentally, multiple load-flow solutions (which has attracted a lot of attention, [10], [11]) can be easily computed because the load-flow procedure is simply reduced to finding the roots of an appropriate polynomial. Thus the $PV$ and $QV$ curves up to the point of collapse can be accurately computed. Moreover, using these polynomial relations, we show that the sensitivity of bus voltage components to the real/reactive loads at that bus can be expressed as a rational function, which is a function of the parameter (real or reactive power) and voltage component. This rational function which characterizes voltage sensitivity can be formulated geometrically as a surface which we call the sensitivity manifold. Graphical exploration of this manifold in the vicinity of the collapse point allows a visualization of the sensitivities near collapse. The rest of this paper is organized as follows. In Sec.II, a brief background and terminology pertinent to the theory of Gröbner basis is presented. The main results are presented in Sec.III along with a discussion, followed by the conclusions in Sec.IV.

II. A BRIEF OVERVIEW OF THE THEORY OF GRÖBNER BASES

A. Overview

The overview described here also appears in [12], but included here for completeness sake. The concept of Gröbner bases (GB) was proposed in [13] to address fundamental problems
in commutative algebra, especially the theory of polynomial ideals. In recent years, owing to computational advances, the concept has found several applications in areas including: systems theory (multivariate polynomial matrix factorization, Bezout identities, state space isomorphisms), signal processing (design of multidimensional FIR and IIR filter banks) and control theory (H∞ theory, Lyapunov functions). One of its important applications arises in the context of solving an algebraic set of polynomial equations of the form:

\[
\begin{align*}
  f_1(x_1, x_2, \ldots, x_n) &= 0 \\
  \vdots \\
  f_n(x_1, x_2, \ldots, x_n) &= 0
\end{align*}
\]

where \( f_1, \ldots, f_n \) are multivariate polynomials in \( n \) indeterminates \( (x_1, \ldots, x_n) \). The GB formalism provides a systematic technique to answer questions such as (i) does the system given by, Eqn.(1), Eqn.(2), \ldots, Eqn.(3) have solutions in the first place? If so, (ii) what is the cardinality of the set of solutions? (iii) when the solution set is finite, can we enumerate and determine all the solutions? Incidentally, it can be noted that questions (i) and (ii) can be answered without having to solve the system. Loosely speaking, the key strategy is to reduce the original set of polynomials to an “equivalent” canonical set of the form:

\[
\begin{align*}
  g_1(x_1) &= 0 \\
  x_2 - g_2(x_1) &= 0 \\
  \vdots \\
  x_n - g_n(x_1) &= 0
\end{align*}
\]

where \( g_1, \ldots, g_n \) are univariate polynomials of \( x_1 \). Note that this canonical set has a structure that resembles the triangular form obtained through Gaussian elimination for linear systems. Because the two systems are equivalent, they possess the same solution set. It is easier to solve the canonical system, since the process involves finding the roots of a univariate polynomial. For example, once all the roots of \( g_1(x_1) = 0 \) are found, then they can be substituted in \( x_2 - g_2(x_1) = 0 \) to find the roots \( x_2 \) and so on, until all the solutions are determined. The computational process of arriving at this canonical set (called the Gröbner Basis) from the original set is accomplished through Buchberger’s algorithm, [13] which has been implemented in numerous symbolic computation packages such as MAPLE, Singular, Mathematica and CoCoA. In what follows, the relevant background and terminology from computational algebraic geometry are introduced to understand the concept of Gröbner Bases and Buchberger’s algorithm.

B. Background and Terminology

Let \( x \) denote an \( n \)-tuple of indeterminates \( (x_1, x_2, \ldots, x_n) \), \( \alpha \) denote an \( n \)-tuple of natural numbers \((\alpha_1, \alpha_2, \ldots, \alpha_n)\) and \( k \) denote a field. Then \( x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n} \) is said to be a monomial. A polynomial \( f \) is then a combination of monomials such that \( f = \sum r_\alpha x^\alpha \) where the coefficients \( r_\alpha \) belong to the field \( k \). The set of all such polynomials denoted by \( \mathcal{R} = k[x_1, \ldots, x_n] \) forms a commutative ring which is referred to as a polynomial ring. A non-empty subset \( I \subset \mathcal{R} \) which is closed under addition and multiplication is called an ideal. More precisely, a subset \( I \subset \mathcal{R} \) is an ideal if (i) \( 0 \in I \) (ii) \( I \) is an additive subgroup of \( \mathcal{R} \). and (iii) \( r \in \mathcal{R} \) and \( i \in I \) implies \( ri \in I \). For example, if \( f_1 \) and \( f_2 \) are polynomials, then the set, denoted by \( I(f_1, f_2) \), of all linear combinations of the form \( h_1 f_1 + h_2 f_2 \) where \( h_1, h_2 \in \mathcal{R} \) forms an ideal which is said to be generated by \( f_1 \) and \( f_2 \) and \( I(f_1, f_2) = \langle f_1, f_2 \rangle \). Alternatively, one can think of \( f_1, f_2 \) as basis elements for the ideal that they generate. A formal way of characterizing the common zeroes of a polynomial system is by introducing the idea of an affine algebraic variety. For example, the set of all common zeroes to the system \( \{ f_1, f_2 \} \) is called the variety corresponding to that system and denoted by \( V(f_1, f_2) \). With the notion of an ideal, we note that \( V(f_1, f_2) = V(I(f_1, f_2)) \). Generalizing this, if \( F = \{ f_1, \ldots, f_n \} \) is a (basis) set of polynomials, then \( \langle F \rangle \) is the set (or the ideal) generated by these basis elements such that \( I = \{ g | g = \sum_i f_i h_i, h_i \in \mathcal{R} \} \). A natural question is to ask does every ideal of the polynomial ring has a finite basis, which is a Hilbert Basis Theorem answers in the affirmative and the ring \( \mathcal{R} \) is said to be Noetherian, (see [14], [15] for details). Thus the problem of finding the zeroes to a system of polynomials \( F = \{ f_1, \ldots, f_n \} \) is equivalent to determining the variety for the ideal generated by that set or \( V(I(f_1, f_2, \ldots, f_n)) \). Since every ideal of a polynomial ring has a finite basis, the question becomes does their exists a basis such that it has the “triangular form” described above? The answer is affirmative for ideals having simple zeroes and this makes it easier to determine the affine algebraic variety. The basis so constructed is called the Gröbner Basis which can be obtained through an algorithmic computation due to B. Buchberger, [13]. This development hinges on two key ideas namely: (a) the concept of monomial orderings and (b) normal form reduction of a polynomial. In the following subsection, a brief introduction to these concepts is provided. For further expository details, one can refer to [16].

C. Monomial Orderings, Normal Forms and S-polynomials

1) Monomial Orderings: The concept of monomial ordering plays a fundamental role in the computation of a Gröbner Basis. Let \( \mathbb{Z}_+^n = \{ \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) | \alpha_i \geq 0, \alpha_i \in \mathbb{Z}, \forall i = 1 \ldots n \} \) denote the set of \( n \) tuples non-negative integers. Then \( > \) is an admissible total ordering on \( \mathbb{Z}_+^n \) if (i) \( \alpha > 0 \ \forall \alpha \in \mathbb{Z}_+^n \) and (ii) \( \forall \alpha, \beta, \gamma \in \mathbb{Z}_+^n, \alpha > \beta \Rightarrow \alpha + \gamma > \beta + \gamma \). Using this notion of ordering, monomials can be ordered in different ways, one of the more popular ones being the lexicographic (lex) order which is used in this paper. Suppose \( \alpha, \beta \in \mathbb{Z}_+^n \), then:

- \( \alpha >_{\text{lex}} \beta \Rightarrow \) the left most non-zero entry in \( \alpha - \beta \) is positive. \textit{Ex.}: If \( \alpha = (4, 3, 5), \beta = (1, 8, 9), \) then, \( \alpha >_{\text{lex}} \beta \) because \( \alpha - \beta = (3, -5, -4) \) and therefore, the monomials are ordered as: \( x^4y^3z^5 >_{\text{lex}} x^4y^3z^5 \).

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Once the monomials within a polynomial are ordered, then the polynomial can be rearranged to reflect that ordering. For example, if \( f_1(x,y,z) = 7xy^3z^2 - 3x^2y^3 + 2x^4y^2z \), then with respect to the lex ordering \( x > y > z \), then the polynomial is reordered in descending order as \( f_1(x,y,z) = -3x^2y^3 + 2x^4y^2z + 7xy^3z^2 \). After such an ordering, one can define the following terms: (i) the multidegree of the polynomial \( f \) as the \( \text{max}(\alpha \in \mathbb{Z}_+^n) \) that appears in \( f \). For example, multideg\( f_1 = (2,3,0) \), (ii) the leading monomial \( \text{LM}(f) \) of \( f \) as \( x^{\text{multideg}(f)} \). In this case, \( \text{LM}(f_1) = x^2y^3z \), (iii) the leading term \( \text{LT}(f) \) of \( f \) as the monomial \( r_{\text{multideg}(f)}x^{\text{multideg}(f)} \). For example, \( \text{LT}(f_1) = -3x^2y^3z \).

In Gaussian elimination, variables are eliminated successively in a certain order to arrive at the reduced triangular form. In the case of polynomial systems, monomial ordering serves to identify the sequence in which the leading terms of a polynomial are eliminated. This aspect will become clear when the concept of normal forms and S-polynomials are introduced, which is the second and most important step in computing the GB.

2) Normal Forms and S-polynomials: Let \( F \) be a finite collection of polynomials. A polynomial \( p \) is said to reduce to another polynomial \( q \) modulo \( F \) if the leading term \( \text{LT}(p) \) of polynomial \( p \) can be deleted from \( p \) by subtracting a certain multiple of a polynomial \( f \) from \( F \) to obtain \( q \). This can be expressed as \( q = p - f \cdot \text{LT}(p) \). This is possible only if the leading term of \( p \) is divisible by the leading term of some polynomial in the set \( F \). In the event this is not possible, i.e. when \( \text{LT}(p) \) is not divisible by any of \( \text{LT}(f) \), \( f \in F \), then \( p \) is said to be irreducible modulo \( F \). Further, if \( p \) reduces to \( q \) modulo \( F \) and \( q \) is irreducible modulo \( F \), then \( q \) is said to be the normal form of \( p \), denoted by \( q = \text{normal form}(p) \).

Example: Let \( F = \{ f_1, f_2 \} \) with \( f_1 = x^2y^2 + x^2y + y \) and \( f_2 = x - y^2 \). Suppose \( p = x^3y^2 + x^3y^4 + y^2 + 7 \). Choose \( f = f_1 \) so that \( \text{LT}(f) = x^2y^2 \) and \( \text{LT}(p) = x^6y^2 \). Then by the computation \( q = p - f \cdot \text{LT}(p) \), yields \( q = -x^6y - x^4y^4 + x^3y + y^2 + 7 \). Note that \( q \) can be further reduced with respect to \( F \) by choosing \( f = f_2 \). This reduction process can be continued until the resultant polynomial \( q \) is no longer reducible modulo \( F \) in which case we say \( q \) is the normal form of \( p \). Also recall that the terms in all the polynomials are ordered with the lex ordering.

With these concepts, it is possible to formulate a definition for a Gröbner Basis. The following is a definition proposed by Buchberger, [13]. Suppose \( F \) is a collection of polynomials, then \( F \) is a Gröbner Basis if all normal forms modulo \( F \) are unique. That is to say, if \( g_2 \) and \( g_2 \) are normal forms of \( g \) modulo \( F \), then \( g_1 = g_2 \).

In order to do compute the Gröbner Basis, we introduce the concept of S-polynomials. These S-polynomials are related to computation of syzygies on the elements of the leading terms of the given polynomials. It is to be noted that the “S” in S-polynomials seems to have originated from syzygy. Given two polynomials \( f \) and \( g \), the S-polynomial of \( f \) and \( g \) denoted by \( \text{spoly}(f,g) \) is defined as:

\[
\text{spoly}(f,g) = \text{LCM}(\text{LM}(f), \text{LM}(g)) \times \left[ \frac{f}{\text{LT}(f)} - \frac{g}{\text{LT}(g)} \right] \quad (8)
\]

The least common multiple or \( \text{LCM} \) of two monomials \( x^\alpha = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_n^{\alpha_n} \) and \( \beta = x_1^{\beta_1} \cdot x_2^{\beta_2} \cdots x_n^{\beta_n} \) is given by

\[
\text{LCM}(x^\alpha, x^\beta) = x_1^{\max(\alpha_1,\beta_1)} \cdot x_2^{\max(\alpha_2,\beta_2)} \cdots x_n^{\max(\alpha_n,\beta_n)} \quad (9)
\]

In essence, \( \text{spoly}(f,g) \) is simply a linear combination of the polynomials \( f \) and \( g \) chosen such that the leading terms cancel.

Example: Let \( f_1 = x^4y^2z^3 + xy \) and \( f_2 = x^3y^4z - x^2y^2z^3 \). Then \( \text{LM}(f_1) = x^4y^2z^3 \) and \( \text{LM}(f_2) = x^3y^4z \) and \( \text{LCM}(\text{LM}(f_1), \text{LM}(f_2)) = x^4y^4z^3 \). Then from Eqn.(8) yields, \( \text{spoly}(f_1,f_2) = x^4y^2z^3 + xy^2 \).

D. Buchberger’s Algorithm

With the concept of S-polynomials, an alternative definition for a GB can be formulated as follows, [13].

Theorem: Suppose \( G = \{ g_1, g_2, \ldots, g_n \} \) be a finite set of polynomials and \( I \) be the ideal generated by \( G \). Then, the following are equivalent (i) \( G \) is a Gröbner Basis, (ii) \( \forall g_i, g_j \in G, \text{spoly}(g_i, g_j) \) reduces to zero modulo \( G \).

The Gröbner basis \( G \) discussed above does not meet the minimality requirement in the sense that any other finite set \( \tilde{G} \subset I \) containing \( G \) is also a Gröbner basis. Thus, the concept of reduced Gröbner basis (RGB) becomes apparent.

The RGB \( G_{\text{red}} = \{ h_1, h_2, \ldots, h_k \} \) for an nontrivial ideal \( I \subset K[x_1, x_2, \ldots, x_n] \) is a unique RGB if it satisfies the following:
(a) The coefficient of all \( \text{LT}(h_i) = 1 \) for all \( i = 1, \ldots, k \).
(b) For any \( h_i \in G_{\text{red}} \), no monomial of \( h_i \) lies in \( \langle \text{LT}(G_{\text{red}} - h_i) \rangle \) for all \( i = 1, \ldots, k \).

While a brief overview of this topic was presented here, the reader is referred to [16] for an algorithm (referred to by Buchberger’s algorithm) and further details.

III. Results

In this section, the proposed approach is illustrated on a three bus system. First, we exemplify the load-flow procedure using Gröbner basis in section Sec.III-A and note some of its features. This is followed by Sec.III-B where the method is extended to analyze static voltage collapse. A discussion of the results obtained in these sections is presented in Sec.III-C.
A. Load Flow Solutions with Gröbner bases

The system we consider is shown in Fig.1, where bus 1 is a slack bus and buses 2,3 are load buses. The system parameters are provided in Appendix I. At bus $j$, let (i) $P_j$, $Q_j$ to be the real/reactive bus power injections and (ii) $V_j = (m_j + jn_j)$ as the rectangular components of the bus voltage, the LFEs for the system can be expressed as,

$$P_j = \sum_{i=1}^{3} G_{ji} (m_i n_j + m_j n_i) + \sum_{i=1}^{3} B_{ji} (m_i n_j - m_j n_i)$$

$$Q_j = \sum_{i=1}^{3} G_{ji} (m_j n_i) - \sum_{i=1}^{3} B_{ji} (m_j n_i + n_i n_j)$$

where $Y_{ij} = G_{ij} + jB_{ij}$ refer to the entries from the system bus admittance matrix, $Y$.

First, we seek to find the RGB for the ideal generated by the set of LFEs, Eqns.(10-11). With $(m_1, n_3, m_2, n_2)$ as the lex ordering, Buchberger’s (see [16] for details) algorithm in MAPLE-10, yields the following set of polynomials for the RGB.

$$\sum_{i=0}^{6} a_i n_2^i = 0$$

$$\sum_{i=0}^{6} b_i n_2^i + A_0 n_2 = 0$$

$$\sum_{i=0}^{6} c_i n_2^i + B_0 n_3 = 0$$

$$\sum_{i=0}^{6} d_i n_2^i + C_0 n_3 = 0$$

For the system parameters and loadings chosen, the coefficients are computed $(a_i, b_i, c_i, d_i, A_0, B_0, C_0)$. The set of zeroes for Eqns.12-15 form the variety for the ideal generated by the LFEs and hence, the elements in this variety constitute all possible solutions to the set of LFEs. In this case, Eqn.(12) is first solved for $n_2$, following which the other variables $m_2, n_3, m_3$ are solved successively from Eqns.(13)-(15) respectively. From the solution set of Eqn.(12) (which can include complex conjugate roots), only the real roots form meaningful solutions to the LFEs. Therefore, the variety restricted to the set of real roots shall be referred to as the admissible variety. In this case, we find four points in the admissible variety, or in other words, four distinct solutions to the LFEs which are tabulated below in Table I.

<table>
<thead>
<tr>
<th>solution</th>
<th>$V_2$</th>
<th>$V_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$(0.2820 - 0.1316j)$</td>
<td>$(0.0082 - 0.0329j)$</td>
</tr>
<tr>
<td>2</td>
<td>$(0.1131 - 0.1238j)$</td>
<td>$(0.0098 - 0.0381j)$</td>
</tr>
<tr>
<td>3</td>
<td>$(0.0067 - 0.0085j)$</td>
<td>$(1.0201 - 0.0640j)$</td>
</tr>
<tr>
<td>4</td>
<td>$(0.0354 + 0.0693j)$</td>
<td>$(0.0069 - 0.0744j)$</td>
</tr>
</tbody>
</table>

Note that the procedure’s ability to extract all possible solutions to the LFEs is simply a consequence of the fact that one can determine all the roots of the polynomials in the RGB for the LFEs. Of course, out of the solutions listed in Table I, solution # 3 corresponds to a meaningful steady state power system operating condition. In the following section, this approach is extended to study static collapse, which is the main objective of this paper.

B. Analysis of Static Collapse

For the analysis in this section, we consider bus # 3 as a $PV$ (with $|V_3| = 1$ p.u.) and bus # 2 as a $PQ$ bus where the real/reactive loadings $P_2, Q_2$ are treated as parameters. Restricting bus # 3 to be a $PV$ bus will simplify one of the LFEs to the form $n_3^2 + n_2^2 = 1$. Later on, we discuss the effect of this simplification on the computational procedure. In what follows, variations on $P_2, Q_2$ are considered one at a time and treated separately.

1) $QV$ analysis at bus # 2: First, we study the effect of smooth variations in reactive power on the voltage at bus # 2. With $P_2$ fixed at 2.0 p.u., $Q_2$ is treated as a parameter and the RGB corresponding to this parameterized set of LFEs is computed which yields the following set of polynomials:

$$f_1(n_2, Q_2) = \sum_{i,j} \alpha_{ij} n_2^i Q_2^j = 0$$

$$f_2(m_2, Q_2) = \sum_{i,j} \beta_{ij} m_2^i Q_2^j = 0$$

For a specified value of $Q_2$, Eqns.(16-17) can be solved for $n_2$ and $m_2$ and hence the bus voltage $V_2$. As the loading $Q_2$ is smoothly varied, the coefficients, and correspondingly the roots, of the polynomials (16-17) smoothly change and therefore, the $QV$ curve at the bus can be determined. Further, the point of collapse corresponds to the minimum value of $Q_2$ for which either of the polynomials $f_1$, or $f_2$ possess no real solutions. The $Q - V$ curve thus determined for the system is shown in Fig.2. To verify this result, the smallest singular value ($\sigma_f$) obtained from a singular value decomposition (SVD) of the load-flow Jacobian is also computed and plotted along the $QV$ curve as shown by the dashed lines. The choice of SVD was motivated from its widespread use in voltage stability calculations, [17], [18], [11] for determining how close the load-flow Jacobian is, to being singular. From Fig. 2, we note that at the point of static collapse ($Q_c = 12.6835$) calculated with Eqns.(16-17), the load-flow Jacobian is singular with $\sigma_f = 0$. 

Fig. 1. Three Bus Example System.
2) PV analysis at bus # 2: Here, the reactive loading \( Q_2 \) is held fixed at 2.0 p.u, and \( P_2 \) is treated as a parameter. The RGB for the resulting set of LFEs is given by:

\[
g_1(n_2, P_2) = \sum_{i,j} \gamma_{ij} n_2^i P_2^j = 0 \tag{18}
\]

\[
g_2(m_2, P_2) = \sum_{i,j} \delta_{ij} m_2^i P_2^j = 0 \tag{19}
\]

The \( PV \) curve at bus # 2 can then be computed similarly and the result is shown in Fig. 3 where the collapse point is observed at \( P_2 = 23.1715 \).

3) Voltage Sensitivity Analysis: The issue of interest is in computing how sensitive the voltage at a bus is, to the real/reactive loading at that bus. The polynomial relation in the bus voltage components \( (m_2, n_2) \) and the loading (say, \( Q_2 \) for example) facilitates the determination of an explicit relation for the sensitivity. Implicit differentiation of Eqns.(16,17) yields,

\[
s_n(m_2, Q_2) = \frac{dn_2}{dQ_2} = \frac{h_n^m(m_2, Q_2)}{h_2^m(m_2, Q_2)} \tag{20}
\]

\[
s_m(m_2, Q_2) = \frac{dm_2}{dQ_2} = \frac{h_n^m(m_2, Q_2)}{h_2^m(m_2, Q_2)} \tag{21}
\]

where, \( h_n^i, h_2^m, i = 1, 2 \) are polynomials in their respective arguments. In the \( (m_2, Q_2) \) plane, the function \( s_m \) can be thought of as a surface which we term as a sensitivity manifold. Evaluating this function for a value of \( m_2 \) that is a load-flow solution yields the numerical value of the sensitivity of that component at that loading. The same observation can also be made in the \( (n_2, Q_2) \) plane with respect to the function \( s_n \). While the surface itself can have local maxima and minima, the sensitivity manifold corresponding to the rational function \( s_m \) also has singularities in the sense that it is not analytic at those points. Some of the singularities of this manifold are interesting from the point of view of static voltage collapse and they correspond to the set of zeroes of the polynomial \( h_2^m \). At the collapse point \( Q_c \) (at which, \( m_2 = m_c \)), the slope \( (dm_2/dQ) \) is unbounded and that corresponds to the singularity of the rational function \( s_m \) at the value \( (0.5, 12.68) \). For illustration, the top and bottom views of the surface \( s_m \) evaluated over a grid \( (0 < Q_2 \leq Q_c), \ (0.45 \leq m_2 \leq 0.55) \) are shown in Figs. 4 and Fig.5 respectively. When the collapse point is approached from below, i.e., \( m_2 < 0.5 \), the slope \( (dm_2/dQ_2) \) is increases rapidly in the positive direction, which is reflected by the sharp peaks. When the collapse point is approached from above, i.e., \( m_2 > 0.5 \), the slope \( (dm_2/dQ_2) \) decreases rapidly in the negative direction, which is reflected by the deep trenches. At all other points, the relatively flat portion of the surface indicates gradual changes in \( m_2 \) for variations in \( Q_2 \).
C. Discussion

One of the limitations with the proposed approach is that the coefficients of the polynomials are large integers. This is partly because (a) the choice of the base quantities for the per-unit system causes the elements of the system admittance matrix to be relatively large compared to actual system parameters and (b) the coefficients of the LFEs are required to be rational numbers, which constrains all entries to the program (MAPLE-10) to be rational numbers. When such entries are processed through Buchberger’s algorithm where the calculations involve remainder arithmetic with polynomials, the resulting coefficients tend to be large integers. Furthermore, we the structure of the LFEs influences the size of these large numbers. For example, we examined the coefficients obtained in the case when bus # 3 is a PQ bus with those obtained when bus # 3 is a PV bus and observed that the number of digits in the coefficients tends to be large integers. Furthermore, we examined two cases: (i) |V_3| = 1.03 and (ii) |V_3| = 1.05. Then the last LFE is represented as:

\[ 100^2n_3^2 + 100^2n_3^2 - 100^2 = 0 \quad \text{and} \quad 20^2n_3^2 + 20^2n_3^2 - 212 = 0 \]

for cases (i) and (ii) respectively. While we do not report the results here (owing to space limitations), we observed that the size of the coefficients is appreciably larger in the former case.

IV. Conclusions

An algebraic-geometric method to analyze static voltage collapse is proposed and illustrated on a simple three bus power system model. The method is based on computing a lex ordered GB for the ideal generated by the set of parameterized LFEs in rectangular coordinates. The method determines all possible solutions for a given set of LFEs in addition to providing explicit polynomial relationships between the bus voltage components and the real or reactive loading at that bus. Using this relation, the PV or QV curves at a bus can be constructed smoothly up to the point of static collapse without having to contend with ill-conditioning issues, or resorting to continuation based methods. Further, it is shown that the sensitivity of the bus voltage components to loadings can be determined and expressed as rational functions which give rise to the sensitivity manifolds. A subset of the zeroes in the denominator of these rational functions correspond to the set of singularities of the manifold which includes the point of collapse. These manifolds are useful in depicting the voltage sensitivities in the vicinity of collapse.

The limitation of the proposed method is the large size of the integers involved in the computations which can pose computational challenges. Thus, Buchberger’s algorithm has overwhelming complexity in the context of application to large power systems. Despite these limitations, the method is appealing because it provides analytical relations between the voltage components and loadings at a bus, which has proved elusive applying conventional techniques. Currently, the authors are working towards alleviating some of these limitations to make the technique amenable to larger systems.

APPENDIX I - System Data

For simplicity, line resistances are neglected and the line reactances expressed in per unit are as follows: \( Z_{12} = j0.04, Z_{13} = j0.03, Z_{23} = j0.025 \) p.u. When bus # 2 is treated as a PQ bus, the loadings are as follows: \( P_2 = 2.5, Q_2 = 1.1, P_3 = 1.3, Q_3 = 0.4 \) p.u and slack bus voltage \( V_1 = 1.05 \). When bus # 3 is treated as a PV bus, \( P_3 = 2, V_3 = 1.0 \) p.u. For the QV analysis at bus # 2, \( P_2 = 4.0 \) p.u and for the PV analysis at bus # 2, \( Q_2 = 2.0 \) p.u.

REFERENCES