DYNAMIC ANALYSIS OF BILINEAR INELASTIC MULTIPLE STORY SHEAR BUILDINGS

by

William E. Saul,* John F. Fleming,** and
Seng-Lip Lee***

ABSTRACT

A method is presented for the analysis of bilinear inelastic multiple story shear buildings subjected to earthquake motions. Relationships are derived expressing the motion at each floor level in terms of the relative deflections of the floor levels with respect to the base of the structure. A numerical integration procedure is used to solve several example problems.

INTRODUCTION

The dynamic analysis of multiple story, multiple bay, structures of an elastic-linear strain hardening, inelastic material, such as shown in Fig. 1, subjected to an earthquake disturbance has been of practical interest for some time and becomes feasible with the advent of readily accessible high speed electronic computers and an accumulation of earthquake records.

The behavior of an elastic-linear strain hardening material, which is elastic or elastic-perfectly plastic in the limits, has been studied for some time. The analysis of members made of such materials under static loading has been treated by several investigators. The dynamic behavior of structures made of such materials, however, has been confined almost wholly to a single degree of freedom system and was recently discussed by the authors (1) in which a partial listing of references can be found. It is recognized, however, that most structures are inadequately described by a single degree of freedom system and, therefore, more recent analyses (2,3) have considered the structure as a discrete multiple degree of freedom system.

One of the more successful and realistic mathematical models of a dynamic building is based on the shear building assumption (4,5) whose restrictions are as follows:

1. Mass is concentrated in a plane at the floor levels;
2. Floors are infinitely rigid and continuous;

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3. The column segment, that is a continuous vertical line of single columns, is massless and fixed to the floor masses.

The definition may be generalized as a building in which the relative displacement between floors is horizontal and the shear force in a story is dependent on only the relative displacement between two adjacent floors. The shear building is in many cases a more realistic model than the space frame for actual buildings (5).

The purpose of this paper is to develop a method for the dynamic analysis of multiple story, multiple bay, buildings of elastic-linear strain hardening materials. Essentially, the n-story building is replaced by an n-degree of freedom lumped mass model based on the shear building assumption and the coupled ordinary differential equations of motion are numerically integrated to obtain a history of the horizontal deflection of the floor masses relative to the base. Compatibility of the deformation is defined by relating the curvature to the deflection at the control points, at the top of a column, through use of the conjugate member analogy described concisely, to include frames and perfectly plastic deformation, by Lee (6) and extended herein to include linear strain hardening inelasticity. The resistance-deflection relationships, resulting from correlating the resisting forces to the compatibility condition through the stress-strain definition, are dependent on the loading history of the structure and also time dependent because of the time dependency of the compatibility equations.

The letter symbols adopted for use in this paper are defined where they first appear.

EQUATIONS OF MOTION

A typical bent of an n-story multiple bay shear building is shown in Fig. 2, and the idealized model of a typical continuous vertical segment of n columns in Fig. 3a. In addition to the shear building assumption discussed previously it is further assumed that all the columns in each story are identical and prismatic.

The equations of motion for lateral movement of the assumed concentrated floor masses of the structure may be described by the relationship

$$\{U\} = [m]\ \{\ddot{y}_T\} \tag{1}$$

where $\{U\}$ are the force resultants at the floor levels, $[m]$ is a diagonal matrix composed of the concentrated floor masses, and $\{y_T\}$, $\{\dot{y}_T\}$, and $\{\ddot{y}_T\}$ are the total lateral deflections, velocities and accelerations of the floor masses. The force resultant at level i can be expressed in the form

$$U_i = F_i(t) - Q_i - C_i \tag{2}$$

where $F_i(t)$ is a time dependent external force, Q_i is the resultant of the shear forces on the floor mass at level i due to all the columns adjacent to the floor, and $C_i = C_i(y_j, \dot{y}_j)$ is a damping force dependent on the con-

struction of the building serving mainly to restrict and decay the vibration. It will be shown later that the force-displacement relationships for the structure can be written in the form

$$\{Q\} = [A]\ \{y\} + \{B\} \tag{3}$$

where A_{ij} is a stiffness influence coefficient and B_i is a residual force vector, both being dependent upon the loading history of the column and the number of columns in the story. Under seismic disturbance, the total acceleration of floor i can be written

$$\ddot{y}_{iT} = \ddot{y}_F + \ddot{y}_i \tag{4}$$

where $\ddot{y}_F$ is the foundation acceleration and $\ddot{y}_i$ the acceleration of floor i relative to the base. For the purpose of this paper internal damping forces will be neglected and external forces at the floor levels assumed zero. In this case Eq. 1, upon the substitution of Eqs. 2, 3 and 4 and rearranging, leads to

$$\left\{ \{W\}\ G\ f(t) - [A]\ \{y\} - \{B\} \right\} dt = [m]\ \{d\dot{y}\} \tag{5}$$

in which

$$\ddot{y}_F = -\ G\ g\ f(t) \tag{6}$$

W_i is the weight of floor mass i, G is a constant called the amplitude factor, g is the gravitational constant, and f(t) describes the time dependence of the seismic disturbance. As will be shown later, Eq. 5 is convenient for numerical solution.

STRUCTURAL ANALYSIS

Moment-Curvature. The stress-strain relationship for an elastic-linear strain hardening material under a uniaxial stress condition is defined in Fig. 1a. The slope E designates a family of parallel lines defining the elastic behavior while $\alpha^2 E$ designates the slope of two parallel lines defining the plastic regime. The bending moment-curvature relationship for a member whose cross section is symmetrical about the plane of bending can be readily derived and and the case of a WF or I section can be accurately approximated by the idealized solid curve given in Fig. 1b. It should be noted that the effect of axial forces is neglected.

For the assumed concentrated mass system the moment-curvature relationships for the control points, located at the top of each column of the story, can be written as

$$\{M\} = \{N\} + [K] \left\{ \{\Phi\} - \{\Psi\} \right\} \tag{7}$$

where

$$K_{ij} = \left[\delta^i\ (\alpha^2 - 1) + 1 \right] k_{ij} \tag{8}$$

In these equations, M_i is the bending moment, Φ_i the curvature at point i, N_i and Ψ_i are the moment and curvature phase constants. The phase constants are either the coordinates of the origin or the last phase change at the point and are constant quantities within each phase. The stiffness is de-

fined by letting $k_{ij} = 0$ if $i \neq j$ and $k_{ij} = EI$ if $i = j$, while the phase coefficient δ^i takes the value $\delta^i = 0$ in the elastic phase and $\delta^i = 1$ in the plastic phase. Therefore, the elements of the stiffness matrix $[K]$ are $K_{ij} = 0$ if $i \neq j$ and $K_{ij} = EI$ or $\alpha^2 EI$ if $i = j$, depending on the phase at point i. The strain hardening coefficient α^2 is that obtained from a simple tension test. The limiting ratios $\alpha^2 = 1$ and $\alpha^2 = 0$ correspond to the elastic and perfectly plastic cases respectively. It will also be helpful for later use to define the following dummy variables

$$\Omega_i = \Phi_i - \Psi_i \tag{9}$$

$$L_i = M_i - N_i \tag{10}$$

where Ω_i and L_i are the curvature and moment phase variables at point i, which are equal to zero at the origin and whenever a phase change occurs at point i.

During a loading cycle, initial conditions must be given and controls used to detect a phase change so as to obtain the phase constants and the stiffnesses at the control points at any time. If the building is initially undisturbed the initial moment-curvature coordinates for any control point are at the origin and, in addition, $\dot{\Phi}_i$ is zero at any point. With zero initial conditions the first phase change will occur if

$$|M_i| > N_i' \tag{11}$$

where N_i' is the positive bending moment at initial yielding of the column at point i as defined in Fig. 1b. The corresponding curvature is denoted Ψ_i'. Whenever the phase at point i is plastic a transition is indicated if

$$\dot{\Phi}_i = 0 \tag{12}$$

Further elastic to plastic phase transitions will occur if

$$|L_i| > 2N_i' \tag{13}$$

and, in addition, the phase variables cannot change signs within a phase. Once a phase transition is found for point i the coordinates, M_i and Φ_i, become the phase constants, N_i and Ψ_i, of that point until the next transition.

<u>Equilibrium</u>. The resisting forces R_i given in Fig. 3b for the model of a vertical column segment are equal to the change in shear at the floor levels as shown in Fig. 3d. A statical analysis yields the resisting force-bending moment relationship

$$\{R\} = 2[H]\{M\} \tag{14}$$

in which

$$H_{ij} = \begin{cases} 1/h_i & i = j \\ -1/h_{i+1} & i = j-1 \\ 0 & \text{otherwise} \end{cases}$$

where the story heights, h_i are not necessarily the same.

The resisting forces Q_i given in Eq. 2 are the resultant of the contributions of all the column shear forces adjacent to a floor mass and can be written

$$Q_i = p^i V_i - p^{i+1} V_{i+1} \tag{15}$$

where p^i is the number of identical columns in the i th story and V_i is the shear force per column. Since the individual resisting forces R_i are related to the shear forces by,

$$R_i = V_i - V_{i+1} \tag{16}$$

the total resisting force can be written as

$$\{Q\} = [P]\ \{R\} \tag{17}$$

where

$$P_{ij} = \begin{cases} p^i & i = j \\ -p^{i+1} + p^i & i < j \\ 0 & i > j \end{cases}$$

Compatibility. The conjugate member of a column in the i th story subjected to lateral forces is shown in Fig. 4a where the conjugate loads, Φ, are plotted on the tension sides and the conjugate moments, y_i and y_{i-1}, are shown in the positive direction. Equilibrium of the conjugate column, observing that the end slopes are zero, yields the recursive equation

$$y_i = y_{i-1} + \Lambda_i + \Delta_i \tag{18}$$

where the phase constant Λ_i is the couple of the curvature area of the last phase change and the phase variable Δ_i is the couple of the increment of the curvature area within the present phase. The latter is depicted in Figs. 4b and 4c for the elastic and plastic phases respectively. In the plastic phase the conjugate load consists of the superposition of two triangular areas defined by two parameters; a variable coefficient a^1 related to depth of penetration of plastic deformation and a constant multiplier α^2 of the continuing elastic contribution. The distance $0.5\ a^1\ h_i$ can be defined as the point at which the change in moment from the preceding plastic to elastic phase transition at control point i is $2N_i'$ can be written

$$a^i = \frac{2N_i'}{M_i - N_i^*} \tag{19}$$

where N_i^* is the moment at the preceding transition. This is determined from similar triangles on the moment diagram. Since the regime under consideration is the following plastic phase the current moment can be written as

$$M_i = N_i^{**} + \alpha^2 \, EI \, \Omega_i \tag{20}$$

where N_i^{**} is the moment at the elastic to plastic phase transition following that at N_i^*. Using these relationships it can be shown that

$$a^i = \frac{1}{1 + \dfrac{\alpha^2 \Omega_i}{2\Psi_i'}} \tag{21}$$

where the velocity has been assumed positive. By defining the dimensionless parameter

$$c^i = q^i \left| \frac{\Omega_i}{\Psi_i'} \right| \tag{22}$$

in which $q^i = 2$ after the first elastic to plastic phase transition after the origin, and $q^i = 1$ thereafter, the parameter a^i can be written for any plastic phase or velocity as

$$a^i = \frac{1}{1 + 0.5 \, \alpha^2 c^i} \tag{23}$$

In a similar manner, the constant multiplier of the continuation of elastic deformation can be shown to be the coefficient of strain hardening α^2 as shown in Fig. 4c.

The couple of the conjugate load Δ_i defined in Eq. 18 and Figs. 4b and 4c can be shown to be

$$\Delta_i = \gamma^i \, \Omega_i \tag{24}$$

where

$$\gamma^i = \frac{h_i^2}{6} \left[\delta^i \, (\alpha^2 - 1) + 1 \right] \left[1 + \delta^i (1 - \alpha^2) \, \frac{(3 + \alpha^2 c^i) \, c^i}{(2 + \alpha^2 \, c^i)^2} \right] \tag{25}$$

in which δ^i and c^i are as previously defined. Expanding the recursive expression given in Eq. 18, the following deflection-curvature relationship can be written:

$$\{y\} = [T] \, \{\Lambda\} + [\Gamma]^{-1} \, \{\Omega\} \tag{26}$$

which, when solved for the curvature, leads to

$$\{\Omega\} = \lceil\Gamma\rceil \{y\} - \lceil\Gamma\rceil \lceil T\rceil \{\Lambda\} \tag{27}$$

where

$$T_{ij} = \begin{cases} 0 & i < j \\ 1 & i \geqslant j \end{cases}$$

$$\Gamma_{ij} = \begin{cases} \dfrac{1}{\gamma^i} & i = j \\ -\dfrac{1}{\gamma^i} & i = j + 1 \\ 0 & \text{otherwise} \end{cases}$$

Equation 25 gives γ^i as a function of Ω_i. Therefore, Eq. 27 contains the variable curvature, Ω_i on both sides of the equation, which necessitates an iterative solution.

RESISTANCE FORCE RELATIONSHIP

The force-deformation relationship previously defined by Eq. 3 can now be derived by relating the equilibrium conditions, described by Eqs. 14 and 17, to the moment-curvature and compatibility conditions given in Eqs. 7, 9 and 27. The coefficients of the resulting force-deformation relationship given in Eq. 3 are

$$\lceil A\rceil = 2\lceil P\rceil \lceil H\rceil \lceil K\rceil \lceil\Gamma\rceil \tag{28}$$

and

$$\{B\} = 2\lceil P\rceil \lceil H\rceil \left\{\{N\} - \lceil K\rceil \lceil\Gamma\rceil \lceil T\rceil \{\Lambda\}\right\} \tag{29}$$

The matrices $\lceil P\rceil$, $\lceil H\rceil$ and $\lceil T\rceil$ are constants for a given building, while the elements of $\lceil K\rceil$, $\{N\}$ and $\{\Lambda\}$ change values whenever a phase transition occurs at any control point. The matrix $\lceil\Gamma\rceil$ is a function of the deflection, consequently, the solution of the equations of motion will require an iterative loop.

NUMERICAL SOLUTION OF EQUATIONS OF MOTION

The equations of motion, Eq. 5, can be solved by using a numerical integration procedure (7) in which the variation in the momentum-force and the deflection-velocity relationships are assumed to be linear over a small time interval Δt. The resulting equations can be written

$$\{U(2)\} = \frac{2}{\Delta t}\,[m]\left\{\{\dot{y}(2)\} - \{\dot{y}(1)\}\right\} - \{U(1)\} \tag{30}$$

$$\{\dot{y}(2)\} = \frac{2}{\Delta t}\left\{\{y(2)\} - \{y(1)\}\right\} - \{\dot{y}(1)\} \tag{31}$$

in which $U_i(1)$ is the resultant of the forces acting on mass point i at the beginning of the time interval and $U_i(2)$ relates to the same resultant force at the end of the interval. The numbers (1) and (2) in all cases refer to the magnitude of their respective variable at the beginning of the interval, t_1, and its value at the end of the time interval, $t_2 = t_1 + \Delta t$, respectively. By combining Eqs. 5, 30 and 31 the equations of motion can be approximated by

$$\left[\frac{4}{(\Delta t)^2}\,[m] + [A(2)]\right]\{y(2)\} = \left[\frac{4}{(\Delta t)^2}\,[m] - [A(1)]\right]\{y(1)\}$$
$$+ \frac{4}{\Delta t}\,[m]\{\dot{y}(1)\} + Gf(1)\{W\} + Gf(2)\{W\} - \{B(1)\} - \{B(2)\} \tag{32}$$

Thus, if the initial conditions $\{y(1)\}$ and $\{\dot{y}(1)\}$ and the matrices $[A(2)]$ and $\{B(2)\}$ are known, values of $\{y(2)\}$ and $\{\dot{y}(2)\}$ can be obtained from Eqs. 31 and 32 respectively. Once these values are known the phase variable curvatures can be obtained from Eq. 27, and the curvature rate from

$$\{\dot{\Phi}\} = \{\dot{\Omega}\} = [\dot{\Gamma}]\,\{\dot{y}\} \tag{33}$$

where $\dot{\Gamma}_{ij}$ results from the differentiation of Eqs. 18 and 24 with respect to time. The resulting equations, in view of Eq. 25 and

$$\frac{\partial \gamma^i}{\partial t} = \dot{\gamma}^i = \frac{h_i^2}{6}\left[\delta^i(\alpha^2-1)+1\right]\left[1+\delta^i(1-\alpha^2)\left\{\frac{3c^i}{2+\alpha^2 c^i} - \frac{2\alpha^2 c^{i\,2}(3+\alpha^2 c^i)}{(2+\alpha^2 c^i)^3}\right\}\right] \tag{34}$$

lead to Eq. 33 in which

$$\dot{\Gamma}_{ij} = \begin{cases} \dfrac{1}{\dot{\gamma}^i} & i = j \\[2ex] -\dfrac{1}{\dot{\gamma}^i} & i = j + 1 \\[2ex] 0 & \text{otherwise} \end{cases}$$

The phase variable moments can now be calculated from

$$\{L\} = [K]\,\{\Omega(2)\} \tag{35}$$

which is obtained by combining Eqs. 7, 9 and 10. The bending moments can them be obtained from Eqs. 10 and 35. Equations 11, 12 or 13 can be used to check for possible phase transitions. Once the values at t_2 have been found, they will become the initial conditions for the next increment of time and the process is repeated.

In the single step forward numerical integration process defined by Eq. 32, no iteration is needed in the elastic regime. In the plastic regime $[A(2)]$ and $\{B(2)\}$ are both functions of $\{y(2)\}$ as a result of the definition of c^i. Since $c^i(2)$ is always larger than $c^i(1)$ a solution can be iterated by projecting an initial value of $c^i(2)$, and ending with a predetermined accuracy between the last two consecutive values of $\{y(2)\}$.

A salient feature of the method of numerical integration used is the ease with which the magnitude of the time increment may be changed at any time. A time increment, defined in terms of the period of the first elastic mode t_n as

$$\Delta t = b_1 \, t_n \tag{36}$$

where $b_1 << 1.0$ can be used for the major portion of the successive integrations. However, whenever a phase transition is detected, its coordinates have most likely been passed. Therefore, the last calculated values should be rejected and the coordinates of the transition pinpointed by using a smaller time increment defined by

$$\Delta t = b_2 \, t_n \tag{37}$$

where $b_2 << b_1$, until the phase transition is again found. The last values of the bending moment and curvature thus obtained are then taken to be the phase constants for the next phase. The stiffness matrix is then recalculated after which the integration can be resumed using the larger time interval. In this study the calculations were performed with an IBM 709 computer.

EXAMPLE PROBLEMS

The solutions obtained for a four-story shear building taken from (5) will serve to illustrate the method of analysis and the effect of various degrees of strain hardening on the relative deflections. The calculations were made for a 4-story single bay frame with the following properties: all lumped floor masses have equal weights of 40 kips; all stories have equal heights of 12 feet; the modulus of elasticity of the material is taken as 30,000 ksi; the initial yield moment is taken to be proportional to the moment of inertia of the column, I_1 is 300 in.^{4}, and I decreases by 50 in.^{4} in each succeeding story to 150 in.^{4} in the fourth floor. The idealized earthquake is a decaying sinusoidal curve described by

$$f(t) = e^{\left(0.1 - \frac{0.4t}{\tau t_n}\right)} \sin \frac{2\pi t}{\tau t_n} \tag{38}$$

where τ, the ratio of the period of the idealized earthquake to the fundamental elastic period of the structure, $t_n = 0.6$ sec., was taken equal to 0.25 and the amplitude factor G equal to 0.3. Thus, the period of the earthquake was 0.15 sec., a realistic figure.

The response curves in Fig. 5 show the lateral deflection of the fourth floor relative to the base for various values of α^2. The plastic regimes of the ends of the columns in each story are indicated below the figure. For values of $\alpha^2 \leq 0.001$ the response curves, for practical purposes, coincide with the one for $\alpha^2 = 0.00001$. Thus, although the formulation becomes singular when $\alpha^2 = 0$ because elements of $[\Gamma]$ become infinite the convergence of the response curves for $\alpha^2 \leq 0.001$ indicates that the solutions for very small values of α^2 approach that for the elastic-perfectly plastic case.

CONCLUSIONS

The example problems illustrate the application of the method of analysis developed herein for determining the dynamic response of elastic-linear strain hardening, inelastic, multiple story, multiple bay, shear buildings to earthquake motions. The method is applicable to structures made of a wide range of materials including elastic and nearly perfectly plastic materials. The method can be carried out readily on a digital computer. No attempt has been made to state any general conclusions regarding the effect of strain hardening on dynamic response.

ACKNOWLEDGMENTS

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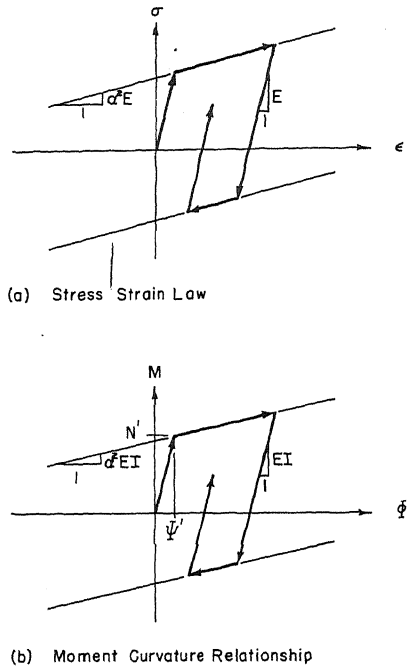


FIG. 1 MATERIAL BEHAVIOR

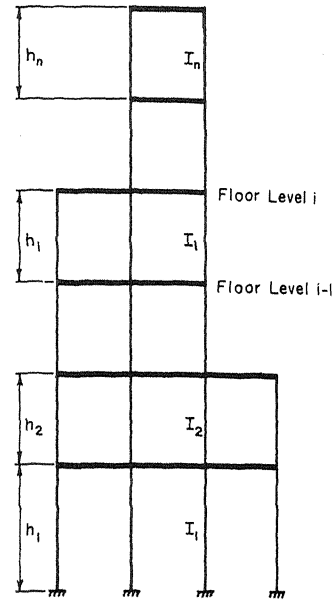
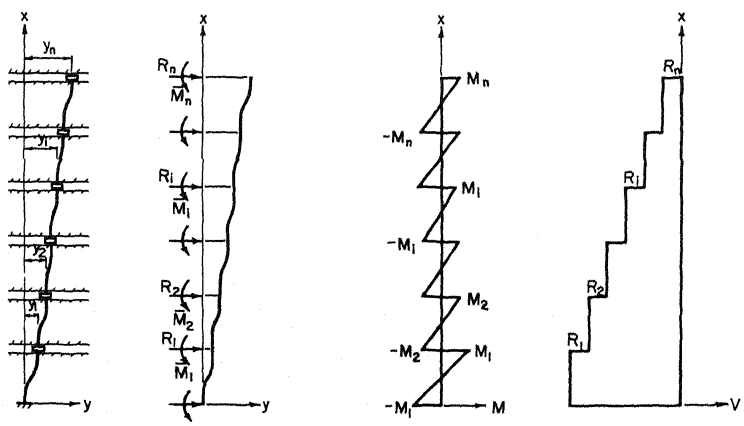
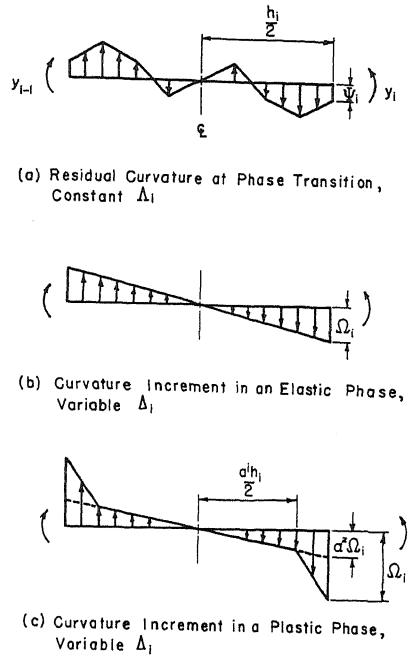


FIG. 2 n-STORY FRAME OF SHEAR BUILDING



(a) Idealized Model of Column (b) Free Body Diagram of Column (c) Bending Moment Diagram (d) Shear Diagram

FIG. 3 COLUMN SEGMENT IN n STORY SHEAR BUILDING



(a) Residual Curvature at Phase Transition, Constant Λ_i

(b) Curvature Increment in an Elastic Phase, Variable Δ_i

(c) Curvature Increment in a Plastic Phase, Variable Δ_i

FIG. 4 CONJUGATE COLUMN

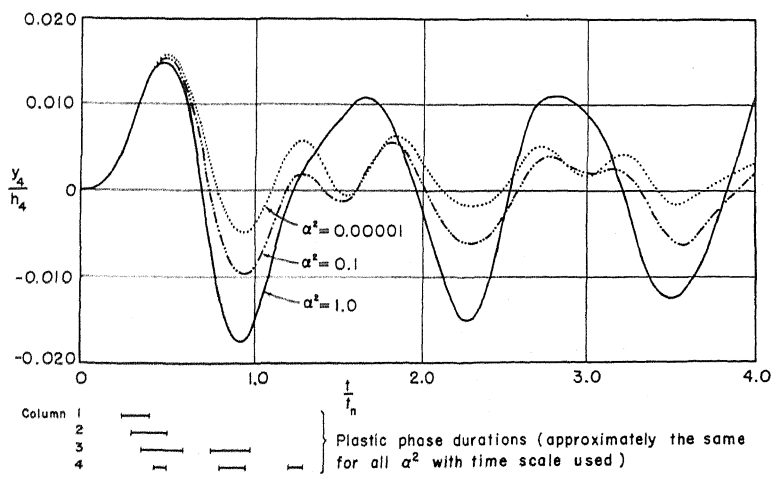


FIG. 5 RESPONSE CURVE FOR FOURTH FLOOR OF A FOUR STORY SHEAR BUILDING.

DYNAMIC ANALYSIS OF BILINEAR INELASTIC MULTIPLE STOREY SHEAR BUILDINGS.

BY: W.E. SAUL, J.F. FLEMING, SENG-LIP LEE

QUESTION BY: A.C. HEIDEBRECHT - CANADA.

In Equation 3, you show a piecewise linear form of the force-displacement relationship, which is derived in the following pages and culminates in Equation 27. This derivation appears to involve non-linear operations in several places, particularly in Equation 25. It is not clear how these non-linearities are arrived at, particularly in the light of the fact that many others have used in a bilinear form of the force-displacement relationship. Please comment on this aspect.

REPLY BY: W. SAUL

In addition, the lateral force - lateral deflection relationship is derived. From Equations 7, 9 and 27, and noting that: $2M_i = hV_i$ (44)

the following is derived

$$\tfrac{1}{2}hV_i = N_i + K_{ij}\left(\Gamma_{jk}y_k - \Gamma_{jk}T_{k\ell}\Lambda_\ell\right) \tag{45}$$

thus, it can be written that

$$V_i = V_{ij}y_j + S_i \tag{46}$$

where

$$V_{ij} = \frac{2}{h}K_{ij}\Gamma_{kj} \tag{47}$$

and

$$S_i = \frac{2}{h}N_i - \frac{2}{h}K_{ij}\Gamma_{jk}T_{k\ell}\Lambda_\ell \tag{48}$$

Therefore, Equation 46 appears to show the V-y relationship to be bilinear also. However, note that the coefficients V_{ij} and S_i are also functions of y_j and, therefore, being nonlinear, the equation requires an iterative mode of solution.

DYNAMIC ANALYSIS OF BILINEAR INELASTIC MULTIPLE STORY SHEAR BUILDINGS

BY W.E. SAUL, J.F. FLEMING AND SENG-LIP LEE

AUTHORS' ADDITIONAL COMMENTS

DERIVATION OF PARAMETRIC VALUES IN PLASTIC INCREMENT OF CURVATURE

In the calculation of the compatibility condition, Eq. 27, the variation in curvature over a column length is required. Whenever the current phase is plastic the variation is piecewise linear and can be given completely in terms of the phase variable curvature Ω_i by defining two dimensionless parameters, a and x as defined in the text and shown in Fig. 6c. Figure 6 illustrates the moment and curvature distribution over half the length of a typical column. Figure 6a is a segment of the M- Φ curve, 6b are moment diagrams and 6c are the corresponding curvature diagrams. Each illustration shows first its values at a phase transition point called point 1 on the M- Φ curve, then an incremental change in value, and finally the values corresponding to point 2 on the M- Φ curve, which are the incremental values added to those at point 1.

Following is a general proof, supplementary to the text, of the parametric values.

Consider the moment diagram, Fig. 6b. Since, by definition, point a corresponds to the point on the column which has undergone a change in moment of 2N' in the same interval during which the control point has changed by the value M-N*, it can be written considering similar triangles that

$$\frac{0.5\ ah}{0.5\ h} = \frac{2N'}{M-N^*} \tag{19}$$

It is also true that

$$N' = EI\ \Psi' \tag{40}$$

and

$$M = N^{**} + \alpha^2 EI\Omega \tag{20}$$

Substitution of Eqs. 40 and 20 into Eq. 19 yields

$$a = \frac{1}{1 + \frac{\alpha^2 \Omega}{2\Psi'}} \tag{21}$$

If point 1 were the origin, the right hand numerator in Eq. 19 would be N'. Also, if the velocity limb were negative, the value of this number, as well as Ω_i, would be negative too. Thus, the definition of c^i (Eq. 22) in

the text allows for these events, and the parameter a is defined (Eq. 23).

Next, consider the curvature increment diagram, Fig. 6c. Similar to the previous argument congruent triangles yield

$$\frac{2\psi'}{a} = \psi^{**} - \psi^{*} + x\Omega \tag{41}$$

In addition,

$$\psi^{**} - \psi^{*} = 2\psi' \tag{42}$$

Thus, substituting Eqs. 42 and 21 into Eq. 41 yields

$$x = \alpha^2 \tag{43}$$

and the proof is complete.

ADDITIONAL RESULTS OF THE EXAMPLE PROBLEMS

Figures 7, 8, 9, and 10 are the presentation of additional results of the example problems described within the text. These results further illustrate the inelastic dynamic response of a strainhardening structure. The coefficient of strainhardening α^2 equals 0.10 in each of the diagrams.

In Fig. 7 there is shown the relative deflection history of the fourth story column. This relationship is more meaningful than the deflection response curve given in Fig. 5 since it pertains to the deformation within that member. From this and the shear building definition the actual shape of the column is known at any time. Comparison of the curve in Fig. 7 with the one for $\alpha^2 = 0.10$ in Fig. 5 shows an interesting parallel.

The moment-curvature response of the fourth story column, in a non-dimensionalized form, is shown in Fig. 8. As noted in the footnote in Fig. 5 there are three significant periods during which the phase is plastic. The period of interruption within the last plastic phase was brief and so does not show up in the footnote. A particularly useful characteristic of this diagram is that it indicates the permanent deformation, or set, retained by the column. Noting this point on the curve its associated relative displacement can be located in the tabulated response data (not included herein). This offset impairs the column's axial load capacity.

Figure 9 shows the deformed shape of a column segment at three different times. From left to right the columns have: 1. all negative relative deflection excepting the first column, 2. half negative, and 3. all positive relative deflection. Thus, they illustrate three possible extremes all significant in terms of sustained stresses because of high relative deflections.

The moment response curve for a column in each of the four stories are given in Fig. 10. The response curve of the fourth floor column is similar to the relative deflection curve in Fig. 7. The phase relation-

ship between these curves shows a time lag associated with the rate at which the disturbance propagates up through the structure. Figures 8 and 10 have a common vertical axis and Figures 5, 7, and 10 are all time dependent plots. By aligning these figures so that similar axis are parallel and noting the footnote in Fig. 5, it is possible to select nearly any desired quantity. Thus, a relative deflection of $(y_4 - y_3)/h \approx 0.0014$ corresponds to the set in this column.

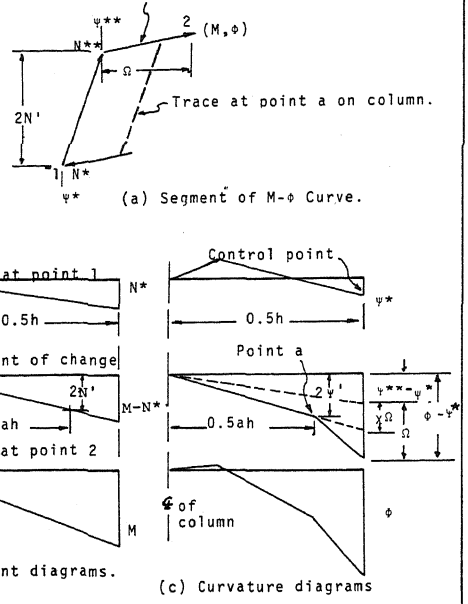


FIG. 6 MOMENT AND CORRESPONDING CURVATURE CHANGE IN COLUMN.

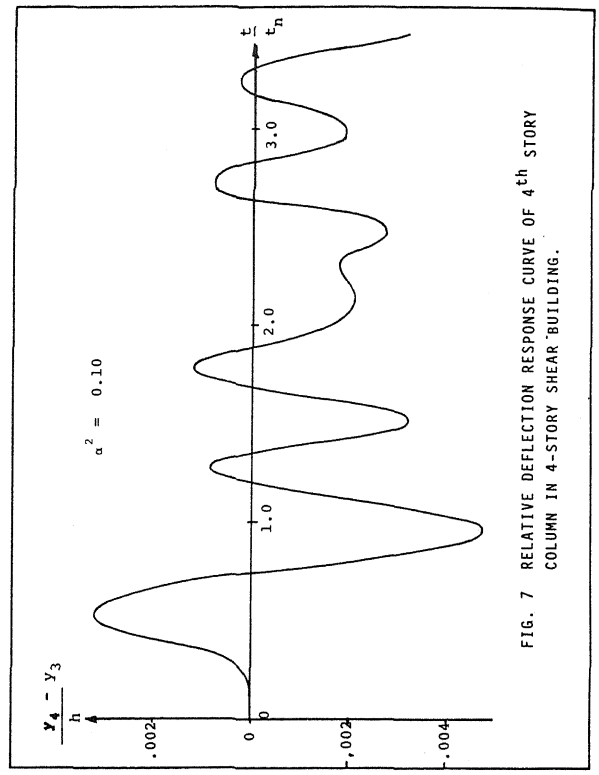


FIG. 7 RELATIVE DEFLECTION RESPONSE CURVE OF 4th STORY COLUMN IN 4-STORY SHEAR BUILDING.

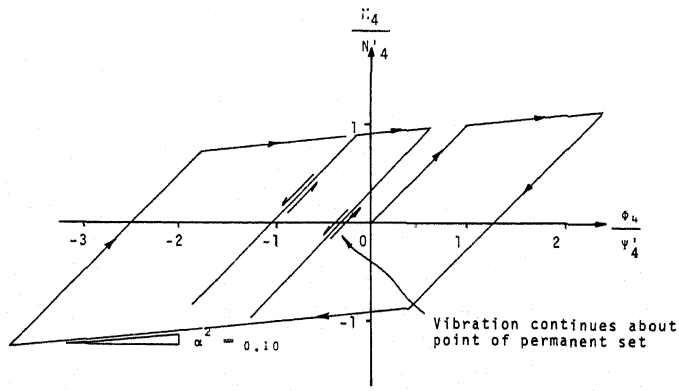


FIG. 8 MOMENT-CURVATURE RESPONSE OF 4th STORY COLUMNS OF 4-STORY SHEAR BUILDING.

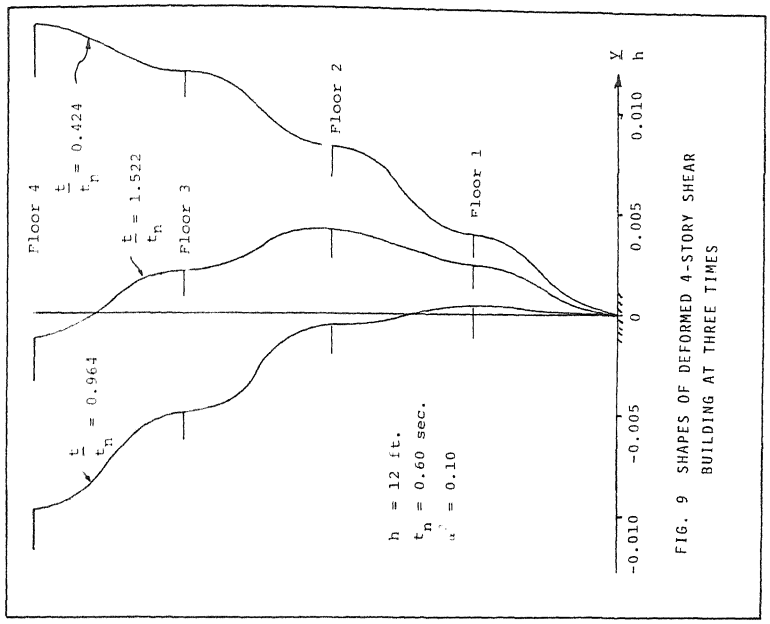


FIG. 9 SHAPES OF DEFORMED 4-STORY SHEAR BUILDING AT THREE TIMES

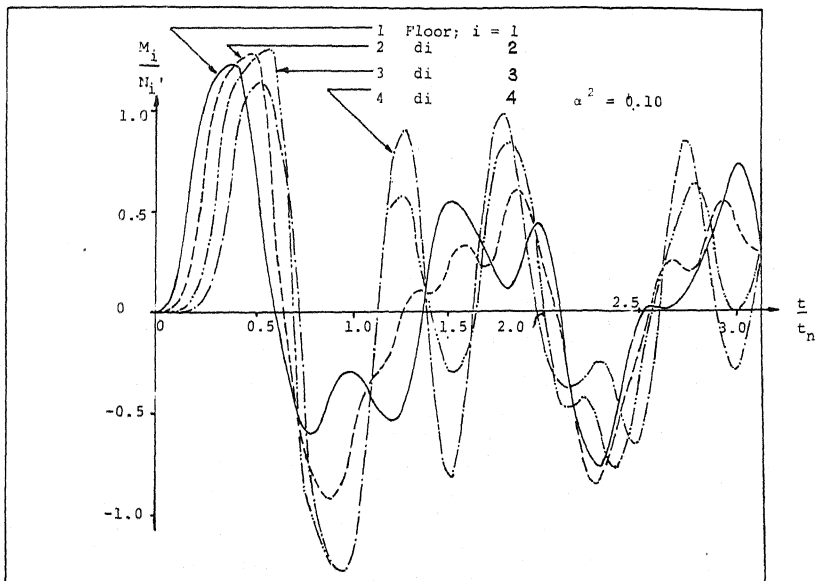


FIG. 10 MOMENT RESPONSE CURVE FOR 4-STORY SHEAR BUILDING