Relations between Elastic/Inelastic Earthquake Response of SDOF Systems by Equivalent Linear Analysis

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SUMMARY:
It is shown that the general tendency of the inelastic earthquake response is interpreted well by the equivalent linear analysis of inelastic systems. The equivalent period is determined at the peak displacement and the equivalent damping is empirically determined from inelastic response analyses as a function of plastic deformation. The relation between the maximum displacement and the yield force obtained from the equivalent linear analysis shows the tendency of the force preservation in the very short period range, the energy preservation in the short period range and the displacement preservation in the long period range. The results of the inelastic response analyses using the simulated earthquake motions are compared with those of the equivalent linear analyses. It is pointed out that the tendency of the relation between the maximum displacement and the yield force is closely related to the tendency of the spectrum shape of the earthquake input.

Key Words: inelastic earthquake response, maximum displacement, equivalent linear analysis

1. INTRODUCTION

Empirical rules about the relations between elastic and inelastic responses of single-degree-of-freedom (SDOF) systems against earthquake motions have long been discussed in the field of earthquake engineering. Housner presented the constant energy concept in the early stage of earthquake response research. Newmark discussed the three types of rules between elastic and inelastic responses, i.e., force preservation, energy preservation and displacement preservation. The rule between elastic and inelastic responses has played an important role when taking account of the effect of inelastic deformation in the earthquake resistant design.

In this paper, it is shown that the general relations between elastic and inelastic earthquake responses can well be interpreted by the equivalent linear analysis of inelastic systems for earthquake motions.

2. EMPIRICAL RELATIONS BETWEEN ELASTC AND INELASTIC RESPONSESES

There have been many studies concerning the empirical relations between the maximum elastic and inelastic responses of single-degree-of-freedom systems against earthquake motions.
Housner proposed a method of limit design by relating the elastic velocity spectrum and the maximum energy by the inelastic deformation in his paper presented in the 1WCEE (Housner 1956).

$$E_p = E_t - E_e, \qquad E_t = \frac{1}{2} m S_V{}^2 \tag{2.1}$$

where E_t is the maximum strain energy in the linear structure, E_e is the elastic strain energy at the yield point and E_p is the plastic energy absorbed by the plastic deformation.

Newmark gave the three different patterns of the relation between the maximum inelastic displacement and the yield force (Newmark 1970). In the short period range of force preservation (constant force), the maximum inelastic displacement increases rapidly compared with the elastic maximum displacement as the yield force decreases. In the medium period range of energy preservation (constant energy), the maximum inelastic displacement increases gradually with the decrease in the yield force, in such a way that the elastic maximum energy is equal to the maximum inelastic energy. In the long period range of displacement preservation (constant displacement), the maximum inelastic displacement remains almost the same as the elastic maximum displacement.

Umemura presented a concept of the equal maximum displacement irrespective of yield level based on the investigation of nonlinear earthquake response of SDOF systems (Umemura 1966).

These empirical rules have been derived from the inelastic response analyses of SDOF systems against recorded earthquake motions in the early stage of research. Few discussions, however, have been done about the causes behind the empirical rules between elastic and inelastic maximum displacements.

3. EVALUATION OF MAXIMUM INELASTC RESPONSE BY EQUIVALENT LINEAR METHOD

The principal assumption of the equivalent linear analysis is that the maximum displacement of the earthquake response of inelastic SDOF system is nearly equal to the maximum displacement of the equivalent linear system having reduced stiffness and increased damping which depend on the degree of the maximum inelastic displacement (Shibata 1975, Shibata & Sozen 1976).

We assume the inelastic force-displacement relation shown in Fig.3.1. The envelope curve is bi-linear having the initial stiffness k and the stiffness αk in the yielding range, where α is the plastic stiffness ratio. Y and N denote the points of the yield displacement δ_Y and the maximum displacement δ_N . The displacement ductility factor δ_N / δ_Y is denoted by μ.

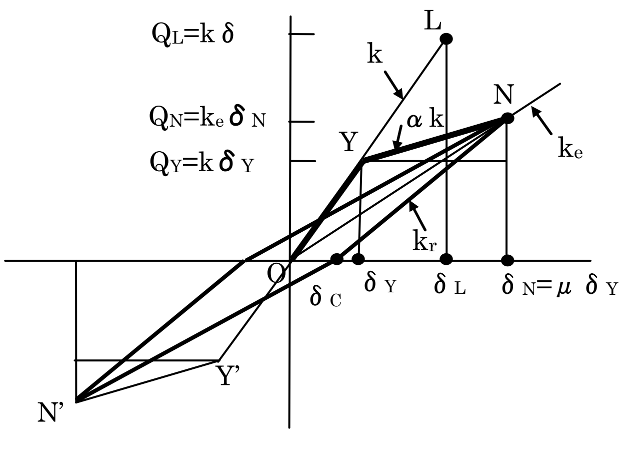


Figure 3.1. Inelastic force-displacement relation

The equivalent stiffness k_e is defined as the secant stiffness at the maximum displacement as follows.

$$k_e = \frac{1+\alpha(\mu-1)}{\mu} k = r_e\, k, \qquad r_e = \frac{1+\alpha(\mu-1)}{\mu} \tag{3.1}$$

where r_e is the equivalent stiffness reduction factor.
The equivalent period T_e at the maximum displacement is given as follows,

$$T_e = \frac{1}{\sqrt{r_e}} T_0 = \sqrt{\frac{\mu}{1+\alpha(\mu-1)}}\, T_0 \qquad T_0 = 2\pi\sqrt{m/k} \tag{3.2}$$

where T_0 is the initial period and m is the mass.

The maximum shear force is expressed using the ductility factor and the plastic stiffness ratio as follows.

$$Q_N = k_e \delta_N = \left(1+\alpha(\mu-1)\right) Q_Y \tag{3.3}$$

where δ_Y and Q_Y are the yield displacement and the yield force, respectively.

The return stiffness k_r from the yielding range is assumed to be the following, which determines the hysteretic damping property of the force-displacement relation.

$$k_r = \sqrt{\frac{1+\alpha(\mu-1)}{\mu}}\, k = \sqrt{r_e}\, k \tag{3.4}$$

The displacementδ_C at the point that the line from the maximum displacement N with the return stiffness k_r intersects the displacement axis is given as follows.

$$\begin{aligned} \delta_C &= \delta_N - Q_N / k_r \\ &= \left(\mu - \sqrt{\mu(1+\alpha(\mu-1))}\right)\delta_Y \end{aligned} \tag{3.5}$$

The hysteretic energy absorbed ΔW and the potential energy W considering the stationary loop with equal displacements in the positive and negative sides are given as follows.

$$\Delta W = 4 \times \frac{1}{2} \delta_C Q_N \tag{3.6}$$

$$W = \frac{1}{2} \delta_N Q_N \tag{3.7}$$

The equivalent damping factor h_e of the stationary hysteretic loop assuming the resonance state is given as follows.

$$\begin{aligned} h_e &= \frac{1}{4\pi} \frac{\Delta W}{W} = \frac{1}{\pi} \frac{\delta_C}{\delta_N} \\ &= \frac{1}{\pi}\left(1 - \frac{\sqrt{1+\alpha(\mu-1)}}{\sqrt{\mu}}\right) = \frac{1}{\pi}\left(1-\sqrt{r_e}\right) \end{aligned} \tag{3.8}$$

Considering the transient nature of the actual inelastic earthquake response, the above equation can be modified as follows including the initial damping factor h_0 of 0.05. This equation corresponds to the damping equation given in the Japanese Building Standard if α is equal to 0. It also conforms to the equation for substitute damping derived from the experiments of R/C frames (Gulkan & Sozen 1974).

$$h_e = 0.25\left(1 - \frac{\sqrt{1+\alpha(\mu-1)}}{\sqrt{\mu}}\right) + 0.05 = 0.25\left(1-\sqrt{r_e}\right) + 0.05 \tag{3.9}$$

Using the equivalent period and the equivalent damping factor, the maximum displacement is obtained as a function of ductility, if the response spectra for any value of period and damping are given, and the corresponding yield shear coefficient is also determined.

The model response spectrum for h=0.05 adopted in this study is the modified safety limit design spectrum in the Japanese building code that has different value in T>4sec as shown in the following.

$$S_A\ (m/s^2) = \begin{cases} 3.2 + 30T & (T \le 0.16s) \\ 8 & (0.16 < T \le 0.64) \\ 5.12/T & (0.64 < T \le 4) \\ 10.24/T^{3/2} & (4 < T) \end{cases} \tag{3.10}$$

The acceleration, velocity and displacement response spectra adopted are shown in Fig. 3.2.

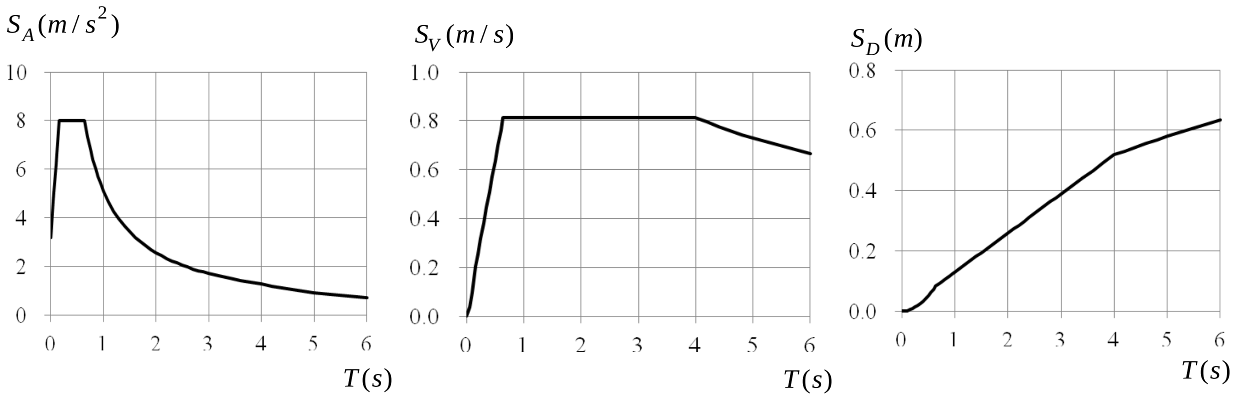


Figure 3.2. Model response spectra

As for the variation of spectral value by damping, the following relation is utilized, which has been adopted for the safety-limit design in the Japanese Building Code.

$$\frac{S(h_e)}{S(0.05)} = \frac{1.5}{1+10h_e} \tag{3.11}$$

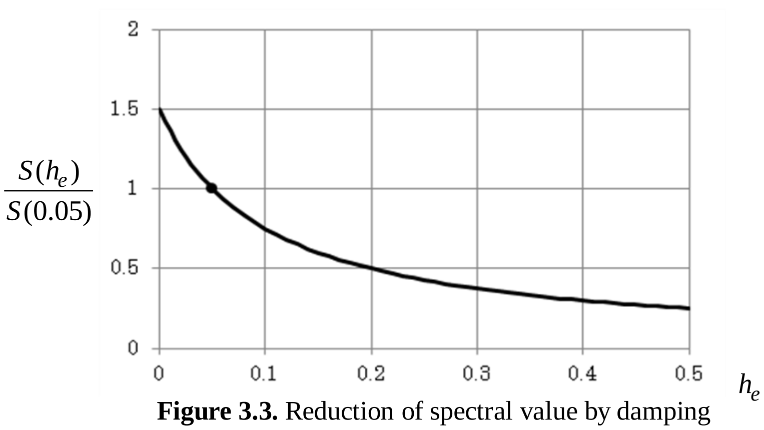


Figure 3.3. Reduction of spectral value by damping

If the value of ductility factor μ is assumed, the corresponding maximum displacement and the maximum force are estimated as follows from Eqns. 3.12 and 3.13.

$$\delta_N = S_D(T_e, h_e) \quad (3.12)$$

$$\frac{Q_N}{m} = \frac{k_e}{m} S_D(T_e, h_e) = \omega_e^{\,2}\, S_D(T_e, h_e) = S_A(T_e, h_e) \quad (3.13)$$

The yield force and the yield displacement corresponding to the assumed ductility factor are determined as follows.

$$\frac{Q_Y}{m} = \frac{1}{1+\alpha(\mu-1)}\frac{Q_N}{m} \quad (3.14)$$

$$\delta_Y = \frac{\delta_N}{\mu} \quad (3.16)$$

By assuming the value of μ gradually increasing from 1 to a certain large value, we can get the relation between the maximum displacement and the yield force using the above equations.
In case the yield force and the yield displacement are given in advance, the ductility factor μ is determined by trial and error so that the assumed ductility factor μand obtained ductility factor δ_N/δ_Y are equal.

4. MAXIMUM DISPLACEMENT VERSUS YIELD FORCE RELATIONS

The general characteristics of the relation between the maximum displacement and the yield force are studied both by the equivalent linear analysis and by the inelastic response analysis.

The inelastic earthquake response analyses are made for the ten waveforms of simulated earthquake motion that are generated to have the response spectrum of Eqn. 3.10. The envelope shape of Eqn. 4.1, proposed by Jennings for the magnitude of about 7 such as El Centro 1940 record, is used (Jennings et. 1969).

$$g(t) = \begin{cases} t^2/16 & t \le 4\,\text{sec} \\ 1.0 & 4 < t \le 15 \\ \exp\left[-0.0924(t-15)\right] & 15 < t \le 30 \\ 0.05 + 0.0005(50-t)^2 & 30 < t \le 50 \end{cases} \quad (4.1)$$

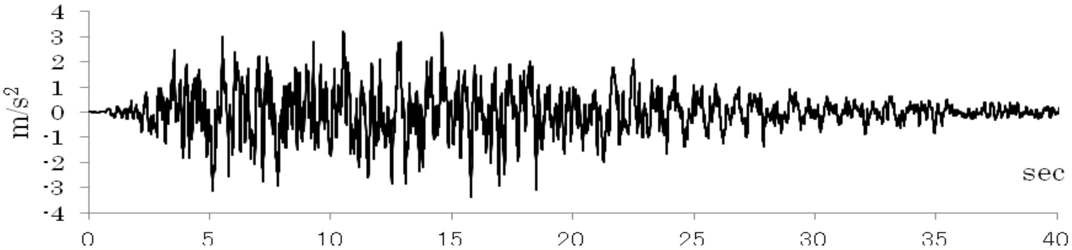


Figure 4.1. A sample waveform of simulated earthquake motion

The inelastic response analyses for six initial periods were conducted by gradually reducing the yield force from 100% to 5% of the elastic response shear. The plastic stiffness ratio α is assumed to be 0.01, which is very close to the elasto-plastic envelope. Degrading hysteresis rules shown in Fig.1 are used.

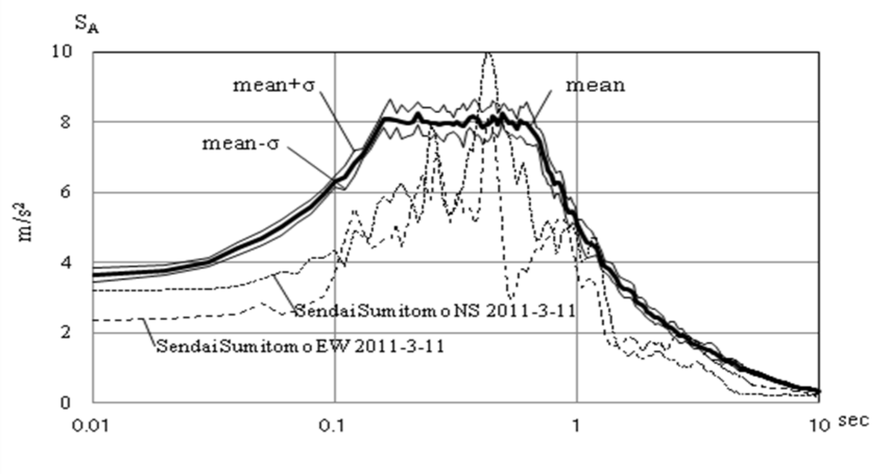
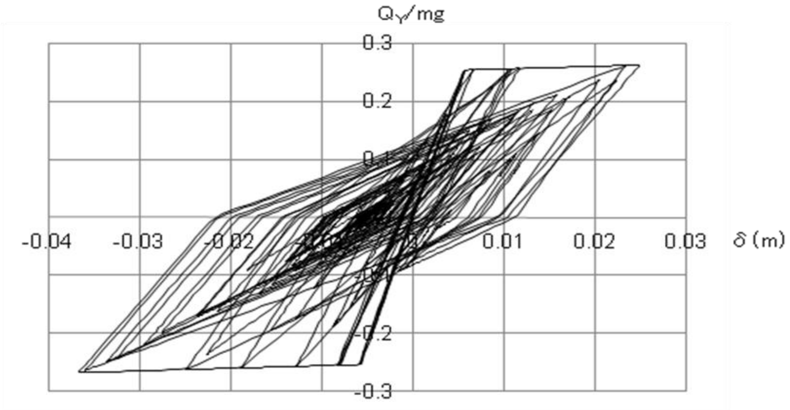
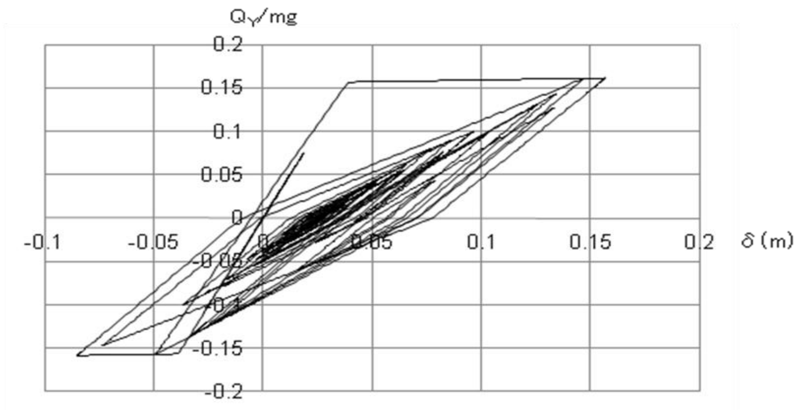


Figure 4.2. Acceleration response spectra of simulated earthquakes (mean plus/minus σ of ten waves) together with those of the recorded motions at Sendai in 2011 Great East Japan earthquake



a) T_0= 0.3sec, Q_Y=0.3Q_L



b) T_0= 1.0sec, Q_Y=0.3Q_L

Figure 4.3. Force - displacement relations of nonlinear SDOF systems

Fig. 4.3. a) shows the response hysteresis curve for the initial natural period of 0.3sec and the yield strength of 30% of linear response force, and Fig. 4.3. b) for the period of 1.0sec and the strength of 30% of linear response force. The former seems to belong to the range of force or energy preservation and the latter to the range of displacement preservation.

The response computations were conducted for ten simulated earthquake waves, and the mean and the standard deviation of the maximum displacement response were determined.

The relations between the maximum displacement δ_N and the yield shear coefficient Q_Y/mg (g: gravity acceleration) obtained by the inelastic response analyses are shown in Fig. 4.5. The abscissa is δ_N and the ordinate is Q_Y/mg. The mean with plus/minus 1σ deviation for the ten simulated earthquake waves is plotted by the symbol ⊢◆⊣ .

The results of the equivalent linear analyses by the procedure in the previous section are also shown in Fig. 4.5. by the thick solid line. The thin straight lines show the relation between the maximum displacement and the yield shear for a given initial period and ductility factors of 1, 2, 4 and 8.

$$\frac{Q_Y}{mg} = \frac{\omega_0^{\ 2}}{\mu g}\delta_N \tag{4.2}$$

The energy preservation rule commonly used to relate the elastic maximum with the inelastic maximum displacement is expressed as follows under the bi-linear force-displacement relation shown in Fig. 4.4. From the assumption that the area OLL' is equal to the area OYNN',

$$\delta_N = \frac{\mu}{\sqrt{2\mu - 1 + \alpha(\mu-1)^2}}\delta_L\ , \quad \delta_L = S_D(T_0, h_0) \tag{4.3}$$

$$\frac{Q_Y}{m} = \frac{1}{\sqrt{2\mu - 1 + \alpha(\mu-1)^2}}\omega_0^{\ 2}\delta_L \tag{4.4}$$

The relation between the maximum displacement and the yield shear coefficient for the case of energy preservation is expressed as follows.

$$\frac{Q_Y}{mg} = \frac{\omega_0^{\ 2}}{g}\left[\delta_N \pm \sqrt{\frac{\delta_N^{\ 2} - \delta_L^{\ 2}}{1-\alpha}}\right] \tag{4.5}$$

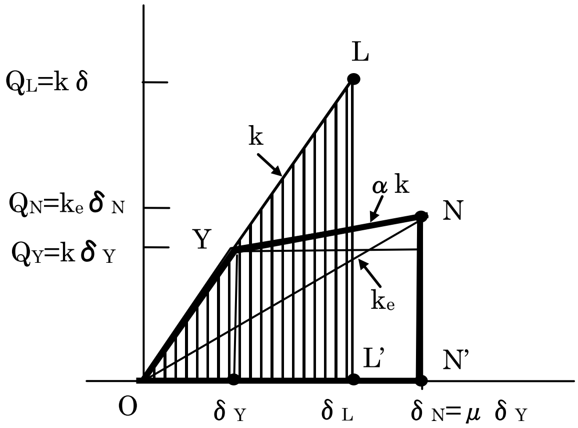


Figure 4.4. Energy preservation rule

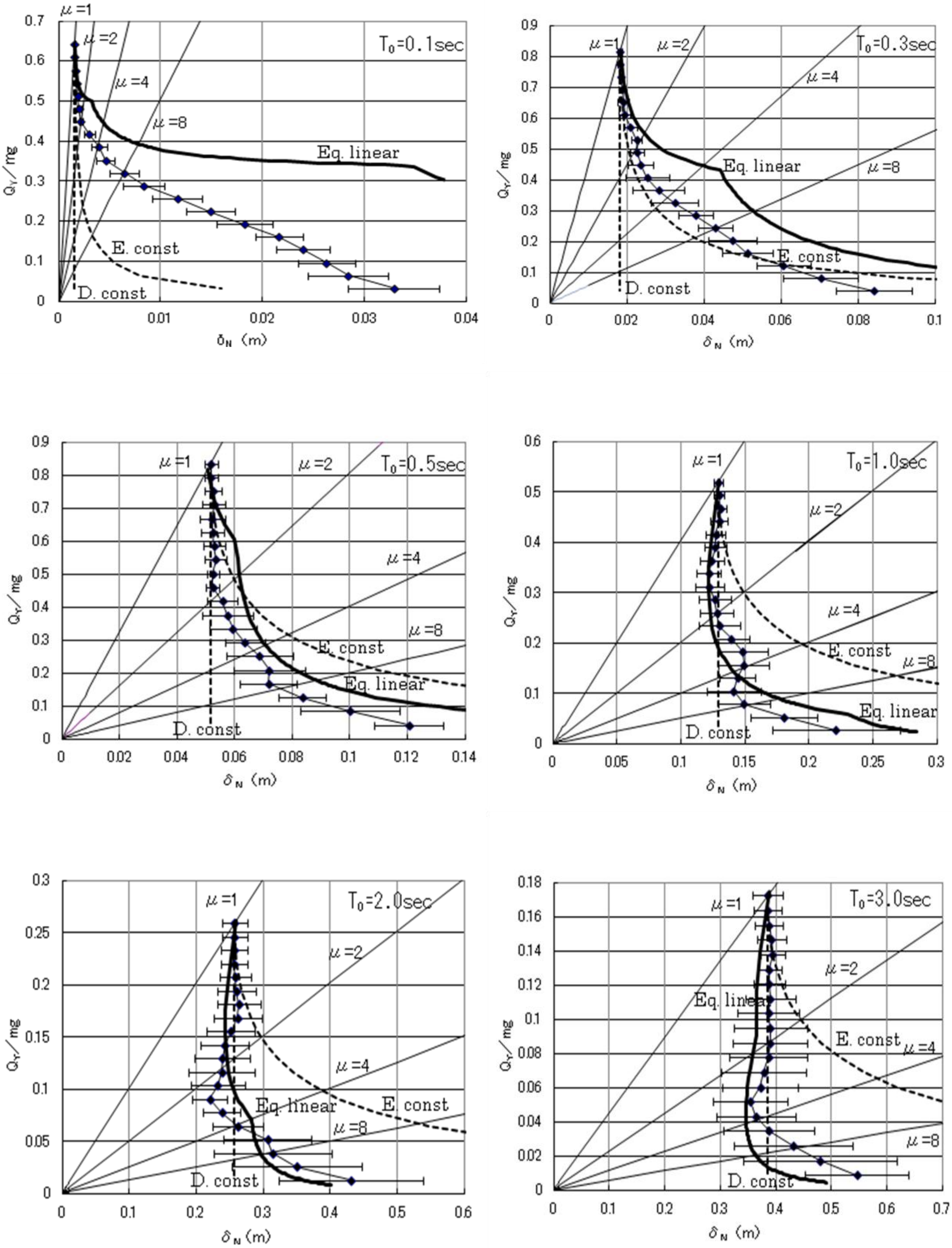


Figure 4.5. Relations between yield force and maximum displacement for various initial periods
(E. const :energy preservation rule, D. const :displacement preservation rule,
Eq. linear :equivalent linear analysis, ├◆┤ :mean plus/minus one standard deviation)

The displacement preservation rule is expressed as follows.

$$\delta_N = \delta_L \ , \quad \delta_L = S_D(T_0, h_0) \tag{4.6}$$

$$\frac{Q_Y}{m} = \frac{\omega_0{}^2 \delta_L}{\mu} \tag{4.7}$$

The relations for the cases of energy preservation and displacement preservation are also shown in Fig. 4.5 by the thin dotted line.

The results from the inelastic response analyses and from the equivalent linear analyses seem to coincide well and show the similar tendencies that are different according to the initial period.

The following tendencies can be seen from Fig. 4.5. concerning the general feature of the relation between the maximum displacement and the yield shear force: 1) the tendency is rather similar to the force preservation rule in the very short period range (increasing spectral acceleration range), 2) the tendency is similar to the energy preservation rule in the short to medium period range (constant spectral acceleration range), 3) the tendency is similar to the displacement preservation rule in the long period range (constant spectral velocity range). This implies that the characteristics of the relations between elastic and inelastic response are closely related to the tendency of the shape of response spectra.

Fig. 4.6. shows the patterns of the displacement - yield force relation obtained from the equivalent linear analysis, in which both the maximum displacement and the yield force are normalized by the values of linear response. It is noted that the curves by the equivalent linear analysis in Fig. 4.5. and in Fig. 4.6. are the same except the operation of normalization. We see that the tendencies of the maximum displacement-yield force relation reflect the shape of response spectrum as the equivalent period shifts due to the increase in plastic deformation caused by the decrease in yield force. The folding points observed in the curves of Fig. 4.6. correspond to the folding points in the model response spectrum shown in Fig. 3.2.

The effects of equivalent damping on the spectral response expressed by Eqns. 3.9 and 3.11 are essential for the approximate evaluation of the inelastic maximum response by equivalent linear method. The general tendencies of inelastic displacement – yield force relation can be well interpreted by the equivalent linear concept using these two equations currently adopted in Japanese Building Code.

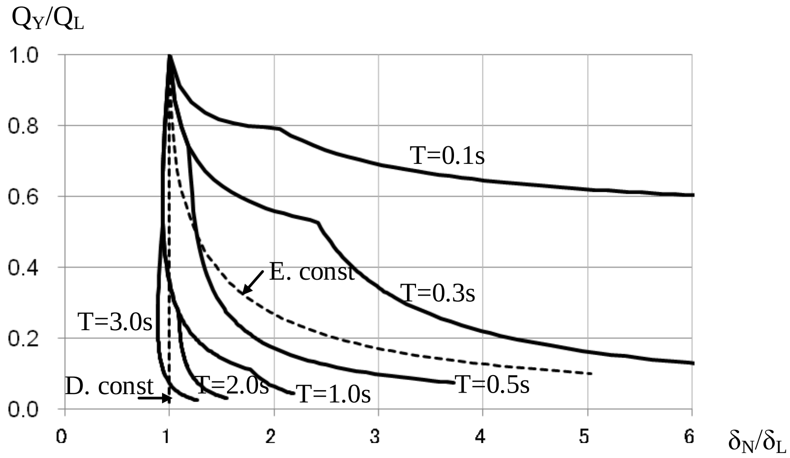


Figure 4.6. Normalized patterns of maximum displacement-yield force relation for different initial periods obtained by equivalent linear analysis

5. CONCLUSIONS

The general characteristics of the relation between the maximum displacement and the yield force in the nonlinear hysteretic earthquake response of SDOF systems are well interpreted in a consistent manner by the concept of equivalent linear analysis.

The empirical rules of force preservation, energy preservation and displacement preservation presented by Newmark can be explained with close relation to the shape of response spectra in short period range (increasing spectral acceleration range), medium period range (constant response acceleration range) and long period range (constant response velocity range), respectively.

Though the inelastic response behavior against earthquake is very complex depending on the transient nature of varying system properties and earthquake input mechanisms throughout the response process, the basic concept of equivalent linear analysis that the maximum inelastic displacement is related to the spectral response of equivalent system at the final maximum state will play an important role in grasping the complicated inelastic earthquake response from the overall and simplified point of view.

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