Numerical Study on Covariance-driven Stochastic Subspace Method in Modal Parameters Identification

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SUMMARY:
The covariance-driven stochastic subspace identification method is a practical way for modal parameters identification under ambient excitation. It was demonstrated in a systematic way that modal parameters were gotten from the matrices obtained from the singular value decomposition, and this process hasn't been explicitly deduced so far. This deduction could do benefits to understanding, application and improvement of the identification method. Numerical results showed that the precision of modal parameters could be improved strikingly by extending matrices, and the frequency of excitation also could be identified when it was a sine or cosine function. This method could be helpful to the identification of excitation source in earthquake engineering.

Keywords: Covariance-driven Stochastic Subspace, Modal Parameters Identification, Frequency Identification of Excitation.

1. INTRODUCTION

Modal parameters identification is an active research direction [1]. Among the identification methods, covariance-driven stochastic subspace is an effective way. It supposed that environmental excitation was white noise. It also supposed that the responses were ergodic and its average value was zero. The principle of covariance-driven stochastic subspace was not demonstrated detailedly. How to get the modal parameters from the matrices obtained from the singular value decomposition is a key problem, which hasn't been proved strictly by now. Some paper even had wrong understanding. This thesis gives the process in a systematic way which was helpful for researcher.

2. MODEL OF COVARIANCE-DRIVEN STOCHASTIC SUBSPACE METHOD

The basic model [2] was introduced

$$\begin{cases} x_{k+1} = A_d x_{k+1} + \omega_k \\ y_k = C_d x_k + \nu_k \end{cases} \tag{2.1}$$

In fact, the characters of ω_k, ν_k were not easy to determine. In order to simplify the problem, it supposed that. ω_k, ν_k were white noises which mean value was zero. The character was written as

$$E\left[\begin{pmatrix} \omega_p \\ \nu_p \end{pmatrix}\begin{pmatrix} \omega_q^T & \nu_q^T \end{pmatrix}\right] = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}\delta_{pq} \tag{2.2}$$

where

$$\delta_{pq} = \begin{cases} 0 & p \neq q \\ 1 & p = q \end{cases} \tag{2.3}$$

Supposed that the initial state x_0 was irrelevant with ω_k and ν_k, then

$$E\left[w_k x_0^T\right]=0\text{，}\quad k=0,1,2,3\cdots \tag{2.4}$$

$$E\left[v_k x_0^T\right]=0\text{，}\quad k=0,1,2,3\cdots \tag{2.5}$$

Through Eq.（2.1）,（2.4）and（2.5）such conclusion could be obtained.

$$E\left[w_{k+i} y_k^T\right]=0\text{，}\quad E\left[v_{k+i} y_k^T\right]=0\text{，}\quad i=1,2,3\cdots \tag{2.6}$$

Eq.（6）showed that w_{k+i} was irrelevant with y_k .Furthermore, it supposed that y_k was ergodic and its expected value was zero[3].

Covariance matrix was written as

$$R_i=E\left[y_{k+i} y_k^T\right] \tag{2.7}$$

The next state of output covariance matrix was written as

$$G=E\left[x_{k+1} y_k^T\right] \tag{2.8}$$

With Eq.（1）and（7）

$$R_i=C_d A_d^{i-1} E\left[x_{k+1} y_k^T\right]=C_d A_d^{i-1} G \tag{2.9}$$

This conclusion described the relationship between R_i and system matrix A_d .In the following section, R_i was made by observation data, and from which the modal parameters could be obtained.

The hankel matrix Y_p was made by responses that could be displacement，velocity and acceleration.

$$Y_p=\frac{1}{\sqrt{N}}\begin{bmatrix} y_0 & y_1 & \cdots & y_{N-1} \\ y_1 & y_2 & \cdots & y_N \\ \vdots & \vdots & \vdots & \vdots \\ y_{i-1} & y_i & \cdots & y_{i+N-2} \end{bmatrix}_{i\times N} \tag{2.10}$$

$$Y_{f1}=\frac{1}{\sqrt{N}}\begin{bmatrix} y_i & y_{i+1} & \cdots & y_{i+N-1} \\ y_{i+1} & y_{i+2} & \cdots & y_{i+N} \\ \vdots & \vdots & \vdots & \vdots \\ y_{2i-1} & y_{2i} & \cdots & y_{2i+N-2} \end{bmatrix}_{i\times N} \tag{2.11}$$

$$Y_{f2}=\frac{1}{\sqrt{N}}\begin{bmatrix} y_{i+1} & y_{i+2} & \cdots & y_{i+N} \\ y_{i+2} & y_{i+3} & \cdots & y_{i+N+1} \\ \vdots & \vdots & \vdots & \vdots \\ y_{2i} & y_{2i+1} & \cdots & y_{2i+N-1} \end{bmatrix}_{i\times N} \tag{2.12}$$

where $i \geq 2n$, and n was the number of system's freedom. Theoretically N was infinite. Supposed that the number of observation data was s

$T_{1|i}$ was written as

$$T_{1|i} = Y_{f1}(Y_p)^T = \frac{1}{N}\begin{bmatrix} y_i & y_{i+1} & \cdots & y_{i+N-1} \\ y_{i+1} & y_{i+2} & \cdots & y_{i+N} \\ \vdots & \vdots & \vdots & \vdots \\ y_{2i-1} & y_{2i} & \cdots & y_{2i+N-2} \end{bmatrix}_{i\times N} \begin{bmatrix} y_0 & y_1 & \cdots & y_{N-1} \\ y_1 & y_2 & \cdots & y_N \\ \vdots & \vdots & \vdots & \vdots \\ y_{i-1} & y_i & \cdots & y_{i+N-2} \end{bmatrix}^T_{N\times i}$$

$$= \frac{1}{N}\begin{bmatrix} \sum_{a=0}^{N-1} y_{a+i}y_a & \sum_{a=1}^{N} y_{a+i-1}y_a & \cdots & \sum_{a=i-1}^{i+N-2} y_{a+1}y_a \\ \sum_{a=0}^{N-1} y_{a+i+1}y_a & \sum_{a=1}^{N} y_{a+i}y_a & \cdots & \sum_{a=i-1}^{i+N-2} y_{a+2}y_a \\ \vdots & \vdots & \vdots & \vdots \\ \sum_{a=0}^{N-1} y_{a+2i-1}y_a & \sum_{a=1}^{N} y_{a+2i-2}y_a & \cdots & \sum_{a=i-1}^{i+N-2} y_{a+i}y_a \end{bmatrix}_{i\times i} \quad (2.13)$$

Because the responses were ergodic, then R_i was written as

$$R_i = E\left[y_{k+i}y_k^T\right] = \frac{1}{N}\sum_{k=0}^{N-1} y_{k+i}y^T{}_k \quad (2.14)$$

With Eq.（13）and Eq.（14）$T_{1|i}$ could be written as

$$T_{1|i} = \begin{bmatrix} R_i & R_{i-1} & \cdots & R_1 \\ R_{i+1} & R_i & \cdots & R_2 \\ \vdots & \vdots & \vdots & \vdots \\ R_{2i-1} & R_{2i-2} & \cdots & R_i \end{bmatrix}_{i\times i} \quad (2.15)$$

In the same way, $T_{2|i+1}$ could be written as

$$T_{2|i+1} = Y_{f2}(Y_p)^T = \begin{bmatrix} R_{i+1} & R_i & \cdots & R_2 \\ R_i & R_{i+1} & \cdots & R_3 \\ \vdots & \vdots & \vdots & \vdots \\ R_{2i} & R_{2i-1} & \cdots & R_{i+1} \end{bmatrix}_{i\times i} \quad (2.16)$$

With Eq.（2.9）, Eq.（2.15）and Eq.（2.16）$T_{1|i}$ and $T_{2|i+1}$ could be written as

$$T_{1|i}=\begin{bmatrix}C_d\\C_dA_d\\\vdots\\C_dA_d^{i-1}\end{bmatrix}_{i\times 2n}\begin{bmatrix}A_d^{i-1}G & \cdots & G\end{bmatrix}_{2n\times i}=O_i\Gamma_i \qquad (2.17)$$

$$T_{2|i+1}=\begin{bmatrix}C_d\\C_dA_d\\\vdots\\C_dA_d^{i-1}\end{bmatrix}_{i\times 2n}A_d\begin{bmatrix}A_d^{i-1}G & \cdots & G\end{bmatrix}_{2n\times i}=O_iA_d\Gamma_i \qquad (2.18)$$

Here definitions of observed matrix O_i and controllability matrix Γ_i were given, [4] which were written as

$$O_i=\begin{bmatrix}C_d\\C_dA_d\\\vdots\\C_dA_d^{i-2}\\C_dA_d^{i-1}\end{bmatrix} \qquad (2.19)$$

$$\Gamma_i=\begin{bmatrix}A_d^{i-1}G & \cdots & A_dG & G\end{bmatrix} \qquad (2.20)$$

For 2n order linear time-invariant systems, the sufficient and necessary condition of controllability and observability was that the rank of O_i and Γ_i was 2n. From the formula of $T_{1|i}$, its rank was also 2n.

Because of observed noises, the rank of $T_{1|i}$ was no less than 2n. Generally singular values caused by noises were far smaller than those caused by true data.

In order to reduce the effects of noises, truncated singular value decomposition was used that made the singular values caused by noises zeros. This was a common signal processing method.

$T_{1|i}$ was written as by singular value decomposition

$$T_{1|i}=USV^T=\begin{bmatrix}U_1 & U_2\end{bmatrix}\begin{bmatrix}S_1 & 0\\0 & S_2\end{bmatrix}\begin{bmatrix}V_1\\V_2\end{bmatrix}^T=U_1S_1V_1^T \qquad (2.21)$$

where U 、V were orthogonal matrix, and $UU^T=U^TU=I$, $VV^T=V^TV=I$, S was diagonal matrix, and $S_1\in R^{2n\times 2n}$.

With Eq.（2.15）and Eq.（2.21）, the following conclusion was obtained.

$$T_{1|i}=O_i\Gamma_i=U_1S_1V_1^T \qquad (2.22)$$

The left of Eq.（2.22） was multiply by $S_1^{-\frac{1}{2}}U_1^T$ and the right by $V_1S_1^{-\frac{1}{2}}$

$$S_1^{-\frac{1}{2}}U_1^T O_i \Gamma_i V_1 S_1^{-\frac{1}{2}} = I \qquad (2.23)$$

where $S_1^{-\frac{1}{2}}U_1^T O_i$ and $\Gamma_i V_1 S_1^{-\frac{1}{2}}$ were non-singular. Supposed that

$$W = S_1^{-\frac{1}{2}}U_1^T O_i \qquad (2.24)$$

The inverse matrix of W was written as

$$W^{-1} = \Gamma_i V_1 S_1^{-\frac{1}{2}} \qquad (2.25)$$

Then the expression of O_i and Γ_i were written as

$$O_i = U_1 S_1^{\frac{1}{2}} W \qquad (2.26)$$

$$\Gamma_i = W^{-1} S_1^{\frac{1}{2}} V_1^T \qquad (2.27)$$

In the same way, T_{2i+1} was written as

$$S_1^{-\frac{1}{2}}U_1^T T_{2i+1} V_1 S_1^{-\frac{1}{2}} = S_1^{-\frac{1}{2}}U_1^T O_i A_d \Gamma_i V_1 S_1^{-\frac{1}{2}} = W A_d W^{-1} \qquad (2.28)$$

Supposed that

$$A_d' = S_1^{-\frac{1}{2}}U_1^T T_{2|i+1} V_1 S_1^{-\frac{1}{2}} = W A_d W^{-1} \qquad (2.29)$$

Because W was unknown, A_d could not be calculated. From Eq.（2.29）, it was known that A_d' could be obtained. Since W was non-singular, A_d was similarity matrix of A_d' .They had the same eigenvalues, from which the frequency could be gotten. Then how to get mode shape was deduced. Supposed that the eigenvectors of A_d and A_d' were Φ_d and $\Phi_d^{'}$ .The relationship between them was written as

$$\Phi_d^{'} = W \Phi_d \qquad (2.30)$$

Supposed that

$$O_i' = U S^{\frac{1}{2}} \qquad (2.31)$$

$$\Gamma_i' = S^{\frac{1}{2}} V^T \qquad (2.32)$$

From the above formula, it was known that O_i' and Γ_i' could be obtained.

From the formula of O_i, it was known that the n

Supposed that the first n rows was C_d' , O_i and O_i' could be written as

$$C_d = C_d' W \qquad (2.33)$$

then

$$C_d' = C_d W^{-1} \qquad (2.34)$$

The conclusion was written as

$$C_d' \Phi_d' = C_d W^{-1} W \Phi_d = C_d \Phi_d \qquad (2.35)$$

where $C_d \Phi_d$ was the mode shape. Then if A_d' was obtained, the frequency and mode shape could be gotten.

3. Numerical experiment

Supposed a three DOF lumped mass system shown in Fig.1 The mass, rigidity and damping of each were 1000kg, 1000 $\mathrm{N/m}$ and 40 $\mathrm{N \cdot s/m}$, Sampling frequency was 50Hz and sampling point was 8000.

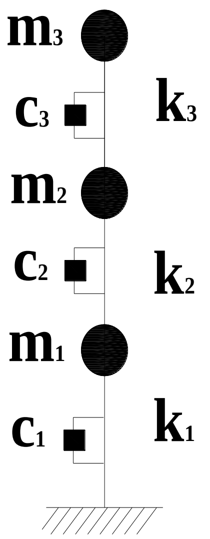

Fig.1 three DOF lumped mass system

The identification results were shown in table 1.

From table 1, it was known that covariance-driven stochastic subspace method could identify frequency accurately, while the accuracy of mode shape was ordinary. Furthermore, the number of Hankel matrix row may affect the accuracy of identification results.

The reasons of error were written as follows:

1. The observed data was limited.

2. The truncated singular value decomposition was used that may eliminate the signal data.

Table 1 identisfication results under different noise

	fundamental frequency （Hz）	second order frequency（Hz）	third order frequency （Hz）	Fundamental mode shape	second order	third order
truth-value	0.445	1.247	1.8019	0.445	-1.247	1.8019
				0.8019	-0.555	-2.247
				1	1	1
no noise	0.4424	1.2386	1.7849	0.4982	-1.2432	0.8633
				0.8407	-0.6093	-1.7644
				1	1	1
relative error	0.58%	0.67%	0.94%	—	—	—
SNR 90	0.4424	1.238	1.7843	0.4972	-1.2439	0.8658
				0.8397	-0.6096	-1.764
				1	1	1
relative error	0.58%	0.72%	0.98%	—	—	—
SNR 80	0.4423	1.2374	1.7826	0.4968	-1.2424	0.8532
				0.8385	-0.609	-1.763
				1	1	1
relative error	0.61%	0.77%	1.07%	—	—	—

4. Conclusion

The parameter identification was an important content of structure damage detection. It could be divided into determinate excitation method and ambient one according to the source of excitation. The parameter identification based on the ambient excitation was practical, since it didn’t need excitation equipment and caused no damage to structures. The principle of the covariance-driven stochastic subspace identification method was deduced in detail. The method used the singular value decomposition technique. How to get the modal parameters from the matrices obtained from the singular value decomposition was a key problem, which hadn’t been proved strictly by now. This thesis gave the process in a systematic way. Numerical results showed that the precision of modal parameters could be improved strikingly by extending matrices, and the frequency of excitation also could be identified when it was a sine or cosine function. Because this hadn’t been proved in theory, the experiment data were not given in this paper.

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